FINITE ELEMENT METHODS
FOR NONLINEAR FREE BOUNDARY PROBLEMS

by

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DEDICATED

TO

MY PARENTS

CERTIFICATE

Certified that the work presented in this thesis, entitled "Finite Element Methods for Nonlinear Free Boundary Problems", by Sri Amiya Kumar Pani has been carried out under my supervision and has not been submitted elsewhere for a degree.

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AKPani

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Amiya Kumar Pani

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CHAPTER 1

INTRODUCTION

1.1 Motivation.

The mathematical modelling of many physically significant problems, such as phase change in chemical and metallurgical processes, or in glass, plastics and oil industries, or in the manufacture of semiconductor materials, leads to boundary value problems involving partial differential equations - especially of parabolic type - with the feature that a part of the boundary is not prescribed in advance, but depends on certain properties of the solution itself. Usually, the dependent variable represents either temperature, or concentration of a solute in a solvent, or pressure of fluid diffusing through a porous medium, or vapour density in an unsaturated mixture. These problems are traditionally called free boundary problems or more classically Stefan problems. An example of such problems is the well-known problem of melting of ice, which was first studied by J. Stefan in early 1889, afterwhom this class of problems is named. The basic difficulty in such problems traces its origin to the determination of a part of the boundary in the process of solution of the problem itself. This naturally leads to a nonlinear problem, which is always harder to solve both analytically and numerically, than the corresponding problem with a fixed boundary.

In the last few years, several numerical techniques like Finite Difference (FD) and Finite Element (FE) Methods are being developed for numerical approximations to solutions of the free boundary problems. However, the finite difference methods have been used extensively for numerical solutions of these problems (see for recent surveys Furzeland [25], Nitsche [48], [51] and Tarzia [59]). But in recent years several finite element methods have been proposed in view of the increased speed and storage capacity of present day Computers. As several of the proposed finite element methods lack sufficient mathematical justification, we consider it worthwhile to present a brief survey of this and make an attempt to justify some of them, under suitable regularity assumptions.

Since the dominant difficulty is introduced by the unknown free boundary, a basic technique used in general is to have a co-ordinate transformation in order to fix the boundary. The penalty one pays for it is the introduction of a highly nonlinear term into the transformed problem. Earlier, Nitsche using this technique initiated the study of finite element error analysis for the single phase linear Stefan problems in one space dimension. The present study is a contribution in the same direction for the nonlinear case. It partially answers the open problem, cited in the workshop on Free Boundary Problems held in Berlin 1977 (see Hoffmann [29, I], pp. 185).

Essentially we are dealing with two types of problems, classified according to the form of the transformed system. In the first case, the transformed problem consists of a nonlinear initial boundary value problem in a fixed domain and two initial value problems in ordinary differential equations (see Chapter 2, 3 and 4), while in the second case the transformed problem is a coupled system of a nonlinear initial boundary value problem and two initial value problems in ordinary differential equations (see Chapter 5). For both the cases the object is to give an error analysis for the Galerkin methods in different norms.

1.2 Notations and Definitions.

In this Section, we introduce some notations and definitions which will be used throughout this work. All functions treated here are real valued functions. For each integer $j \geq 0$, $D^j f = \dfrac{\partial^j f}{\partial x^j}$.

For each $t \geq 0$, let $\Omega(t) \subset R$ be a bounded domain. On the space of square integrable functions on $\Omega(t)$, the L^2-inner product and norm are denoted by

$$(u,v) = \int_{\Omega(t)} uv \, dx$$

and

$$||u||^2 = (u,u) \text{ respectively.}$$

As usual, $L^\infty(\Omega(t))$ shall denote the space of bounded measurable functions on $\Omega(t)$ and

$$||u||_{L^\infty(\Omega(t))} \equiv \operatorname{ess}\sup_{x\in\Omega(t)} |u(x,t)| < \infty.$$

The support of a function f : Supp $(f) \equiv \overline{\{x : f(x) \neq 0\}}$.

For any nonnegative integer α, $C^\alpha(\Omega(t))$ denotes the space of functions with continuous derivatives upto and including order α in $\Omega(t)$, As usual, $C_0^m(\Omega(t))$ denotes all $C^m(\Omega(t))$ functions having compact support in $\Omega(t)$, while $C_0^\infty(\Omega(t))$ is the space of all infinitely differentiable function with compact support in $\Omega(t)$.

For each non-negative integer m, the Sobolov space of order m on $\Omega(t)$, denoted by $H^m(\Omega(t))$ (respectively, $H_0^m(\Omega(t))$, is defined as the closure of $C^m(\Omega(t))$ (respectively, $C_0^\infty(\Omega(t))$ with respect to the norm

$$||u||^2_{H^m(\Omega(t))} = \sum_{j=0}^{m} \int_{\Omega(t)} |D^j u(x,t)|^2 \, dx.$$

Equivalently, $H^m(\Omega(t))$ is the space of all functions whose distributional derivatives of order less than or equal to m are in $L^2(\Omega(t))$. Clearly, $L^2(\Omega(t)) \equiv H^0(\Omega(t))$. $H^m(\Omega(t))$ is a Hilbert space with inner product defined by

$$(u,v)_{H^m(\Omega(t))} = \sum_{j=0}^{m} \int_{\Omega(t)} D^j u \, D^j v \, dx.$$

for a complete discussion of Sobolev spaces, see Adams [1].

We also define $W^{m,\infty}(\Omega(t))$ for each $t \geq 0$ as the closure of $C^{\infty}(\Omega(t))$ with respect to the norm

$$||u||_{W^{m,\infty}(\Omega(t))} = \sum_{j=0}^{m} ||D^j u||_{L^{\infty}(\Omega(t))} .$$

In case $I = (0,1) = \Omega(t)$, then we shall omit I from $H^m(I)$, $H_0^m(I)$, $L^{\infty}(I)$ and $W^{m,\infty}(I)$ and the norm in $H^m(I)$ will be denoted by $||\cdot||_m$.

If X is a normed linear space with norm $||\cdot||_X$ and $\varphi : (a,b) \to X$, then we denote by

$$||\varphi||_{W^{k,q}(a,b;X)} = \Big[\sum_{\beta=0}^{k} ||\frac{\partial^{\beta}\varphi}{\partial t^{\beta}}||^q_{L^q(a,b;X)} \Big]^{1/q}, \quad 1 \leq q < \infty$$

and $||\varphi||_{W^{k,\infty}(a,b;X)}$ is accordingly defined.

In case $(a,b) = (0,T)$ and $X = H^m$ or $W^{m,\infty}$, we write

$||\varphi||_{W^{k,q}(H^m)}$ for $||\varphi||_{W^{k,q}(0,T;H^m(I))}$ or $||\varphi||_{W^{k,q}(W^{m,\infty})}$

for $||\varphi||_{W^{k,q}(0,T;W^{m,\infty}(I))}$.

For convenience, we use $\varphi_x = \frac{\partial \varphi}{\partial x}$, $\varphi_{xx} = \frac{\partial^2 \varphi}{\partial x^2}$, $\varphi_t = \frac{\partial \varphi}{\partial t}$ and $\varphi(1) = \varphi(1,t)$, if $\varphi = \varphi(x,t)$. Throughout this work, K will always denote a generic constant which may or may not be same. On occasion, we will show that a constant depends on certain parameters while being independent of others.

1.3 Mathematical Preliminaries.

Consider the following parabolic equation in a fixed domain $I \times (0,T]$,

(1.3.1) $$-u_t + Lu = f, \quad (x,t) \in I \times (0,T],$$

with either Dirichlet boundary conditions

(1.3.2) $$u(0,t) = u(1,t) = 0, \quad t > 0$$

or mixed boundary conditions

(1.3.2′) $$u_x(0,t) = u(1,t) = 0, \quad t > 0$$

and appropriate initial condition,

where L is a second order (possibly nonlinear) elliptic operat

Weak formulation. Let $\overset{0}{H}{}^2(I)$ be the space of all functions $\varphi \in H^2(I)$ satisfying either

$$\varphi(0) = \varphi(1) = 0$$

or

$$\varphi_x(0) = \varphi(1) = 0.$$

Now the weak solution u of (1.3.1) and (1.3.2) or (1.3.2′) is defined as follows : Find a function $u : [0,T] \to \overset{0}{H}{}^2(I)$ such that

$u_t \in H^1(I)$ in the sense of distribution for each t >

u satisfies the initial condition,

and

$$(1.3.3) \qquad (u_{tx}, v_x) + (Lu, v_{xx}) = (f, v_{xx}),\ v \in \overset{\circ}{H}{}^2(I),\ t > 0.$$

Let us now turn to the formulation of Galerkin method for approximating the solution u of (1.3.3).

Galerkin procedure. Let $\overset{\circ}{S}_h$ denote a finite dimensional subspa of $\overset{\circ}{H}{}^2(I)$. Then a continuous time Galerkin approximation of th solution u is defined by a function $u^h : [0,T] \to \overset{\circ}{S}_h$ such that

$$(1.3.4) \quad (u^h_{tx}, \chi_x) + (Lu^h, \chi_{xx}) = (f, \chi_{xx}),\ \chi \in \overset{\circ}{S}_h,\ 0 < t \leq T$$

with a suitably chosen initial function $u^h(x,0)$.

Let v_i, $i = 1, 2, \ldots, N$ be a basis for the finite dimensional space $\overset{\circ}{S}_h$. Then the Galerkin approximation $u^h \in \overset{\circ}{S}_h$ can be written as

$$(1.3.5) \qquad u^h = \sum_{i=1}^{N} \alpha_i(t) v_i(x).$$

Setting the above expression of u^h and taking $\chi = v_j$, $j = 1, 2,$ in (1.3.4), we obtain an initial value problem for the system of N ordinary differential equations. These ordinary differen tial equations in α_i's are nonlinear if atleast one of L or f is nonlinear in u. Thus, the determination of u^h amounts to solving the system of N differential equations in α_i's with an approximate initial condition. This set of equations can be solved by the variety of numerical methods available for solution of a system of ODE, one of which is to discretize it in time and solve a system of difference equations. In the

sequel, we shall call this the fully discrete approximation.

Approximating finite dimensional spaces. Generally the finite dimensional space $\overset{o}{S}_h$, considered to approximate u in the Galerkin procedure, belongs to a regular $\overset{o}{S}{}_h^{r,2}$ family and satisfies certain inverse properties. We recall here the definitions of $S_h^{r,k}$ families as described in Babuska and Aziz [5], Oden and Reddy [56].

Definition 1.3.1. A class of finite dimensional subspaces S_h, $0 < h \leq 1$ is referred to as an $S_h^{r,k}$ family if and only if the following conditions are satisfied :

(i) $S_h^{r,k} \subset H^k$, $r+1 > k \geq 0$.

(ii) $P_r \subset S_h^{r,k}$, where P_r be the space of polynomials of degree $\leq r$, for regular refinements of a finite element mesh over the prescribed domain.

(iii) For each fixed $h \in (0,1]$ and for each $v \in H^m$, there exists a constant K_o, independent of h and v such that for $0 \leq j \leq \min(m,k)$ and $r \geq 0$

$$\inf_{X \in S_h^{r,k}} \|v-X\|_j \leq K_o h^\sigma \|v\|_m, \tag{1.3.6}$$

where $\sigma = \min(r+1-j, m-j)$.

For example of such spaces when $k = 2$, one can refer to Ciarlet [13], Oden and Reddy [56], Douglas et al. [20] and Bramble and Schatz [12]. $S_h^{r,k}$ space contained in $\overset{o}{H}{}^2$ will be called $\overset{o}{S}{}_h^{r,k}$ spaces.

Definition 1.3.2. (Inverse property). A family of finite dimensional subspaces $S_h^{r,k}$ is said to possess the inverse property if there exists a positive constant K_o such that for all $j \leq k$,

$$(1.3.7) \qquad ||X||_k \leq K_o\, h^{-(k-j)} ||X||_j \,, \text{ for } X \in S_h^{r,k} .$$

In the one dimensional case, we shall sometimes refer to the following as the inverse property (it will be clear from the cases)

$$(1.3.8) \qquad ||X||_{W^{j,\infty}} \leq K_o\, h^{-1/2} ||X||_j ,\ 0 \leq j < k,$$

for every $X \in S_h^{r,k}$.

The assumption that such a property holds for all elements in $S_h^{r,k}$ can be found in Nitsche [47] and Ciarlet [13]. From time to time, we assume one of the following inverse properties for the space $\overset{o}{S}_h$, which belongs to a regular $\overset{o}{S}{}_h^{r,2}$ family

$$(1.3.9) \qquad ||X_{xx}|| \leq K_o\, h^{-1} ||X_x||$$

and

$$(1.3.10) \qquad ||X_x||_{L^\infty} \leq K_o\, h^{-1/2}\, ||X_x|| , \text{ for } X \in \overset{o}{S}_h .$$

We are interested in finding the error $u-u^h$ in terms of h. This estimate measures the closeness of u^h to u in suitable norms. If we get an error estimate for the Galerkin

approximation in H^m-norm with the same power of h that is present in the approximation property of the finite element space $\overset{\circ}{S}_h$, we call the error estimate an optimal estimate.

In the following we recall a well-known Theorem for establishing the existence and uniqueness for operator equation satisfying coercivity [13].

Theorem 1.3.3. (Lax-Milgram Theorem). Let V be a Hilbert space with norm $||\cdot||$ and $A(\cdot,\cdot) : V \times V \rightarrow R$ be a bilinear form with the following properties :

(i) boundedness of $A(\cdot,\cdot)$: $|A(u,v)| \leq M||u|| \; ||v||$

(ii) V-coercivity of $A(\cdot,\cdot)$: $A(u,u) \geq \alpha \; ||u||^2$, $\alpha > 0, u, v \in V$.

Then for any continuous linear functional f on V, there exists a unique solution u_o in V such that

$$A(u_o, v) = f(v), \text{ for all } v \in V.$$

We now state some results concerning the regularity of solution to a general class of elliptic boundary value problems of the form, (see Oden and Reddy [56]) :

$$\begin{aligned} Lu &= f \quad \text{in } I, \\ B_k u &= 0 \quad \text{on } \partial I, \end{aligned} \qquad (1.3.11)$$

where $Lu = \sum_{i,j=0}^{m} (-1)^i D^i (a_{ij} \; D^j u)$,

$$B_k u = \sum_{j \leq q_k} b_{kj} \; D^j u \;, \qquad 0 \leq q_k \leq 2m-1$$

and f is a given function.

Let L^* be the adjoint of L and $A(u,v)$ be the bilinear form associated with the elliptic operator L. Consider $D(L) = \{\varphi \in H^m(I) : B_k\varphi = 0, \ x \in \partial I\}$

For $v \in D(L)$,

$$A(u,v) = (Lu,v) = (u,L^*v).$$

Assume further that (i) $A(u,v)$ is bounded, that is $|A(u,v)| \leq M||u||_m||v||_m$,

(ii) $A(\cdot,\cdot)$ is strongly H^m-coercive that is $A(u,u) \geq \alpha||u||_m^2$, $\alpha > 0$ and $u,v \in H^m$.

Now we state the following regularity Theorem, see Oden and Reddy [56], Necas [46] and Agmon [2].

<u>Theorem</u> 1.3.4. If $f \in H^k$, $k \geq 0$ and $A(\cdot,\cdot)$ satisfies the above two properties, then the solution u of

$$(1.3.12) \qquad A(u,v) = (f,v), \text{ for every } v \in D(L)$$

belongs to H^{2m+k} and there is a constant C_k, independent of u, but depending on m such that

$$(1.3.13) \qquad ||u||_{2m+k} \leq C_k||f||_k.$$

<u>Some inequalities</u>. Below, we present without proof inequalities which will be of frequent use in error analysis in subsequent chapters.

<u>Proposition</u> 1.3.5. If $\varphi \in H^1(I)$ and φ vanishes at any point $x' \in [0,1]$, then

$$(1.3.14) \qquad ||\varphi|| \leq ||\varphi_x||$$

and

$$(1.3.15) \qquad |\varphi(x)| \leq \sqrt{2}||\varphi||^{1/2}||\varphi_x||^{1/2}, \text{ for } x \neq x' \in [0,1].$$

This is one dimensional Poincáre inequality and some times known as Rayleigh-Ritz inequality, see Hardy et al. [28].

Proposition 1.3.6. Let $\varphi \in H^1(I)$, then

$$(1.3.16) \qquad ||\varphi||_{L^\infty(I)} \leq ||\varphi|| + \sqrt{2}\, ||\varphi||^{1/2}||\varphi_x||^{1/2}.$$

Let us recall here the Sobolev imbedding theorem, the proof of which can be found in Adams [1], Oden and Reddy [56].

Proposition 1.3.7. Let Ω be a bounded and smooth domain in R^n. Let j and m be non-negative integers and let p satisfy $1 \leq p < \infty$. Then for $mp > n$, an imbedding of the form

$$W^{m+j,p}(\Omega) \to W^{j,q}(\Omega), \quad p \leq q \leq \infty.$$

holds that is there is a constant $C = C(m,n,p,q;\Omega)$ such that

$$||\varphi||_{W^{j,q}(\Omega)} \leq C||\varphi||_{W^{m+j,p}(\Omega)}, \quad p \leq q \leq \infty \text{ and where}$$

$W^{m,p}(\Omega)$ and $||\cdot||_{W^{m,p}(\Omega)}$ are the Sobolev space and the norm respectively, for detailed description refer to Adams [1].

In one dimensional case, the following proposition is easy to establish.

<u>Proposition</u> 1.3.8. Let $\varphi \in H^1(I)$; then

(1.3.17) $$||\varphi||_{L^\infty} \leq C||\varphi||_{H^1}.$$

Further, if φ vanishes atleast at one point $x' \in [0,1]$, then

(1.3.18) $$||\varphi||_{L^\infty} \leq ||\varphi_x||.$$

We shall also use the following inequalities. For more detail, see Hardy et al. [28].

I. $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$, for $a,b \geq 0$ and $\varepsilon > 0$.

II. $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, for all $a,b \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

(Young's inequality).

In integral form, if f, g are both real valued and $f \in L^p$, $g \in L^q$, then

$$\int_I fg \leq \frac{1}{p} ||f||^p_{L^p} + \frac{1}{q} ||g||^q_{L^q}, \text{ for } \frac{1}{p} + \frac{1}{q} = 1 \text{ and } p > 1.$$

III. (generalization of Young's inequality).

$$abc \leq \frac{a^p}{p} + \frac{b^q}{q} + \frac{c^r}{r}, \text{ for all } a,b,c \geq 0, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \text{ and } p \geq 1$$

Similarly in the integral form, for real valued functions f,g,h if $f \in L^p$, $g \in L^q$ and $h \in L^r$, then

$$\int_I fgh \leq \frac{1}{p} ||f||^p_{L^p} + \frac{1}{q} ||g||^q_{L^q} + \frac{1}{r} ||h||^r_{L^r}, \frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1, p > 1.$$

IV. $\int_I fg \leq ||f||_{L^p}||g||_{L^q}$ for $f \in L^p$, $g \in L^q$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ (Hardy's inequality).

For $p = q = 2$, this inequality is known as Schwarz's inequality.

We state without proof, the following Lemma. For a proof, see Halanay [27, pp. 7].

<u>Lemma</u> 1.3.9 (Gronwall's Lemma). Let f, g and h be piecewise continuous non-negative functions defined on an interval $a \leq t \leq b$, g being non-decreasing. If for each $t \in [a,b]$

$$f(t) + h(t) \leq g(t) + \int_a^t f(t')\, dt' ,$$

then

$$f(t) + h(t) \leq g(t) \exp (t-a).$$

The following is a discrete analogue of Gronwall's Lemma, see Lee [39].

<u>Lemma</u> 1.3.10 (Discrete Gronwall's Lemma). Let $f(t)$, $g(t)$ and $h(t)$ be non-negative functions, defined on $\Gamma_\Delta = \{t \in [o,T] : t = j\Delta t,\ j = o,1,\ldots,N;\ N\Delta t = T\}$, $g(t)$ being nondecreasing. If

$$f(t) + h(t) \leq g(t) + C\Delta t \sum_{\tau=o}^{t-\Delta t} f(\tau),$$

where C is a positive constant, then

$$f(t) + h(t) \leq g(t) \exp (Ct).$$

We close this Section by stating the well-known Schauder fixed point theorem. For reference, see Istrǎtescu [32].

Let X be a Banach space and $\mathrm{Dom}(f)$ = Domain of f.

Theorem 1.3.11 (Schauder fixed point theorem). Let C be a bounded closed convex nonempty subset of X. Let

$$f : C \subset \mathrm{Dom}(f) \to C$$

be completely continuous. Then there exists atleast one $u_o \in C$ such that

$$f(u_o) = u_o.$$

1.4 A Brief Survey.

For the last several years, the free boundary problems, particularly Stefan problems have received steady attention on account of the emergence of new areas of applications providing significant impetus to their study. This is quite evident from the fact that during the last decade many international conferences were devoted exclusively to this area of mathematics. Below, we present briefly a survey of finite element methods for numerical solution of free boundary problems. Several detailed surveys on numerical solution of free boundary problems, especially Stefan problems are available due to Nitsche [48], [51], Furzeland [25], Meyer[43], Elliot and Ockendon[22], and Tarzia [59]. However, the presentation and direction in the present survey is more geared to the type of equations, we are dealing with.

The techniques, used for numerical solution of the free boundary problems can conveniently be grouped into 'front tracking methods', where the position of the free boundary is

predicted along with the solution of the governing equations and 'fixed domain methods', where the reformulated governing equations are solved over a fixed domain and the position of the free boundary is determined a posteriori.

Front tracking methods. The front tracking methods require that the phase front evolves smoothly in time and space, and one usuall has to draw on some a priori knowledge of the solution based on physical model in order to judge whether the front is trackable. Of course, in a given application such decisions are usually easy to make. Let us now look at some specific front tracking methods

Bonnerot and Jamet [7], [8] proposed a finite element front tracking method in which they discretized the domain by means of isoparametric space-time finite elements. The free boundary is approximated by a polygonal line whose vertices coincide with triangulation nodes. In this method the change of the free boundary is tracked continuously, while the elements deform continuously. It appears from the numerical experiments that the method is of second order accuracy. Recently [9], they have extended to two dimensional problems, but the order of accuracy is one instead of two. In an attempt to improve the accuracy of the method, Bonnerot and Jamet [10] proposed a similar space-time finite element method for one dimensional Stefan problem with higher degree (biquadratic, instead of bilinear) elements. It provides approximations which are continuous in space variable, but admit discontinuities with

respect to time variable at each time step. The accuracy is seen to be of third order, as seen from numerical experiments, although no mathematical proof to that effect is available. In addition to accuracy, this method is specially appropriate for the computation of the solution, which admits singularities at the initial time or on the boundary. Further, in their work in 1981 they [11] extend and modify this third order accurate front tracking method in order to solve one-dimensional multiphase Stefan problems with appearing and disappearing phases. The numerical results show that the free boundaries start smoothly with a vanishing initial speed which has not been mathematically proved.

An analogous method, suggested by Lynch and O'Neill [40] incorporates a continuous mesh deforming scheme directly in the approximating finite element equations to take into account the displacement due to the change of the free boundary. The basic difference between this and Bonnerot-Jamet method lies in the fact that the present method is based on the Galerkin approximation in space and uses finite difference in time, where as the other involves space-time Galerkin approximation. By allowing continuous mesh deformation, as dictated by the boundary condition, the free boundaries lie on element boundaries. This circumvents the difficulties inherent in interpolation of parameters and dependent variables across the regions where those quantities change abruptly.

Mori [44] suggested a lumped mass finite element method, based on time dependent basis functions for solving one phase Stefan problem in one space dimension. The stability and convergence of the method has been studied without any error estimates and some numerical examples are provided to illustrate the method. This idea is carried over to two phase problem and to problems in higher space dimension, see Mori [45]. Like all the above methods, the mathematical results concerning the rate of convergence for the present one are still not worked out.

Alexander et al. [3] recently applied the moving finite element method with piecewise linear elements on triangles with moving nodes to the Stefan problem in enthalpy form. Rubinsky and Cravalho [57] also presented a one dimensional finite element method in which the grid is stationary and a moving node that coincides with the position of the change of phase interface is tracked continuously in time.

Jaisuk and Rubinsky [33] developed a numerical method involving finite elements for the solution of multidimensional heat transfer problem in the presence of phase change. The basic idea of the method is as follows : The energy equation in the media and the energy balance equation on the change of phase interface are taken as independent governing equations and are solved by finite element formulation, yielding the temperature distribution in the media as well as the continuous displacement of the interface.

The applicability of the aforementioned space-time finite element method of Bonnerot and Jamet is limited by the requirement that the jump condition must be of explicit type. In more than one space dimension, their strategy of determining the movement of the free boundary meets with difficulties. Following their idea of discretization of the weak formulation with isoparametric space-time finite elements, Hsien Li [30] has proposed quite recently a method starting from a weak enthalpy formulation. It avoids explicit treatment of the jump condition on the free boundary so that the computation is simplified and the method is more flexible. It can deal with conditions more general than those which are encountered in the enthalpy method. Let us remark here that although rigorous error estimates have not been done yet, an analysis for a one dimensional two phase Stefan problem does indicate that the discretization scheme with biquadratic elements is convergent and unconditionally stable. A second order convergence can be expected under some smoothness assumptions on the solution and on the free boundary.

Let us now turn to the method of lines for the finite element solution of the free boundary problems. This is based on reducing the problem in one space dimension to a sequence of ordinary differential equations with free boundaries. Finn and Varoğlu [24] developed a finite element technique, based on this to solve a one dimensional single phase Stefan problem. It employs a functional with variable

domain of integration. However, this work does not contain anything dealing with the question of the rate of convergence. There is a need for further research in this direction.

In [60] Wellford and Ayer discussed a space-time discontinuous finite element method, introduced earlier by Wellford and Oden [61] for a multiphase heat conduction problem. The method uses a fixed grid of standard space-time finite elements in conjunction with certain special space-time finite elements including the free boundaries. The finite element model on special elements is defined by discontinuous interpolation. The resulting model is a true free boundary formulation in that the jump in the heat flux and the position of the free boundary are dependent variables. Only numerical results are discussed without the rate of convergence.

Fixed domain method. Lack of smoothness, required for a front tracking method makes it difficult to apply in certain situations. The alternative is to essentially ignore the free boundary position by solving a reformulated problem over a fixed domain.

The most popular recent fixed domain method uses a reformulation of the free boundary problem as a variational or quasivariational inequality. This approach is still under active development. In the following, we give some recent publications in this directions.

Ichikawa and Kikuchi [31], [37] are concerned with the formulation and finite element analysis of one or two phase Stefan problems by freezing index, which is obtained by special transformation of the temperature field. The resulting variational inequality has been discretized into a system of linear inequalities, which can be solved by optimization techniqueslike the projectional successive over-relaxation (SOR) method. The convergence of this method has been checked numerically. In an attempt to present error analysis Oden and Kikuchi [55] considered a finite element approximation of a one-phase Stefan problem encountered in the study of thawing of a frozen media. For implicit finite difference in time and the linear finite elements in space, a second order convergence in L^2-norm is established.

Elliot [21] using finite element methods has expressed the elliptic variational inequalities, resulting from the implicit time discretization of the weak formulation of the two-phase Stefan problem in terms of a quadratic programming problem, which can be solved using projectional SOR or conjugate gradient methods. The SOR method is shown to be globally convergent. There is a good survey in this regard in Elliot and Ockendon [22]. More abstract theorems on the convergence of the finite element solution of parabolic free boundary problems can be found in Jerome [34] and Gastaldi [26].

Another important version of the fixed domain method is the so called enthalpy formulation for the Stefan problem in conductive heat transfer with change of phase. Here the equation is written as

$$(1.4.1) \qquad \nabla.(k\nabla u) - \frac{\partial H(u)}{\partial t} = q ,$$

where $H(u)$ is a monotone function of u with a jump discontinuity at the phase change temperature (usually taken as $u = 0$). The numerical solution of (1.4.1) is reasonably well understood. One can for example, smoothen $H(u)$ and apply standard solution techniques for the resulting nonlinear diffusion equation. Smearing the jump of H over a non-zero range $(-\varepsilon, \varepsilon)$ amounts to a physical model where the substance undergoes a phase transition in this zone rather than at a fixed temperature $u = 0$. The free boundary is identified with the level set $u = 0$, available from the solution of (1.4.1). The first numerical work in the two phase Stefan problem based on enthalpy formulation of which we are aware was the constructive existential analysis of Kamenomotskaja [36]. It was based on an explicit finite difference scheme. Another work of the same nature is the result of Milton Rose. During 1960's there appeared other papers modelled on Kamenomotskaja viz. Samarskii and Moiseenko, Budak, Soboleva and Uspenskii. One must mention here the work of Meyer [42] along with that of Solomon and Lazaridis in this connection. For a recent survey, see [43] and [59].

In [35], Jerome and Rose have derived the rate of convergence for regularizations of the multidimensional two phase Stefan problem with homogeneous Neumann boundary condition and used the regularized problems in enthalpy form to define a backward difference scheme in time and Galerkin approximation in space over a C^0-piecewise linear spline space. They have established an L^2-convergence of order $\sqrt{\varepsilon}$ for the ε-regularization and an L^2-rate of convergence of order $(h^2/\varepsilon + \Delta t/\sqrt{\varepsilon})$ for the Galerkin estimates. This leads to the natural choices of $\varepsilon \sim h^{4/3}$, $\Delta t \sim h^{4/3}$ and a resulting $O(h^{2/3})$ L^2-rate of convergence of the numerical scheme for the weak solution of the enthalpy equation. An essentially $O(h)$ rate is demonstrated when $\varepsilon = 0$ and $\Delta t \sim h^2$ in the Galerkin scheme under a boundedness hypothesis on the Galerkin approximations. The latter result is consistent with the computational experience.

Yet another approach to (1.4.1) is to use $u(H)$ in the enthalpy formulation leading to equations

$$\frac{\partial \varphi}{\partial t} = \Delta A(\varphi) + Q \,, \tag{1.4.2}$$

where $u = A\varphi = H^{-1}(\varphi)$ and $Q = \dot{q} \circ H^{-1}$. The phase transition now takes place at $u = 0$, but the so called mushy zone appears at least in the numerical solution of (1.4.2), where both phases coexist. Ciavaldini [14] discretized the weak form of (1.4.2) by a quadrature rule and solved the resulting problem using both explicit and implicit finite element schemes.

Convergence proofs for the explicit finite element schemes were given by Ciavaldini and for implicit schemes by Schäfer in Hoffman [29 , III]. Recently Epperson [23], based on the weak form of (1.4.2) developed a finite element method for a nonlinear Stefan problem. Semidiscrete error estimates have been established assuming only Lipschitz regularity of the nonlinearity. The estimates obtained here are of lower order 1/2 as opposed to 2/3 in [35] .

The last fixed domain method in which our interest lies can be treated at the same time as a front tracking method because the free boundary explicitely enters into the differential equation. This is based on a co-ordinate transformation to reduce the problem to one in a fixed domain. The penalty one pays for it is the introduction of a highly nonlinear term into the transformed problem. This requires considerable regularity of the solution. The idea of co-ordinate transformation to fix the free boundary was first proposed by Landau [38] and first used numerically for a finite difference approach by Crank, see Ockendon and Hodgkin [54]. Since its initial introduction, the idea has been used widely in differing numerical techniques to solve the resulting fixed domain problems. For example Budak et al., Wentcel, and Uspensky used finite difference schemes for nonlinear Stefan problems both for single phase and multiphase situations. For references, see Tarzia [59].

Based on this, Nitsche introduced in a series of significant papers [48] - [53] the study of error analysis for

the finite element Galerkin approximations to the single phase linear parabolic free boundary problems in one space dimension. In [48], a semi-discrete Galerkin method for a single phase Stefan problem was proposed and analyzed. The optimal error estimates were also derived. In [50] - [51], the error analysis was further extended to cover the case such as those which occur in oxygen diffusion problems. Nitsche [51] also discussed some refinements and further generalizations, especially related to the Problem III in Magenes [41].

1.5 Outline of the Dissertation.

The layout of the thesis is as follows : In the second chapter, we deal with a single phase non-linear Stefan problem with Neumann boundary condition and the same equations with Dirichlet boundary conditions forms the object of study of the third chapter. In the second chapter optimal rates of convergence in H^1 and H^2-norms are established for continuous time Galerkin approximations, where as in the third chapter optimal error estimates are deduced for continuous as well as discrete time Galerkin approximations. In these as well as the two subsequent chapters, the approach has been to straighten the free boundary by a suitable co-ordinate transformation and then to apply H^1-Galerkin procedures for approximating the solution. Further, the global existence of the Galerkin approximations has been established by fixed point arguments.

In the fourth chapter, a single phase Stefan problem with quasilinear parabolic equation in non-divergence form is considered and a priori estimates in L^2, H^1 as well as H^2 norms are derived.

Chapter five describes both fully and semidiscrete H^1-Galerkin approximations for a semilinear Stefan problem with a nonlinear source term. Unlike in the previous chapters, this problem leads to a system of coupled equations. Therefore, a more careful treatment is necessary. Optimal rates of convergence in L^2, H^1, $W^{1,\infty}$ and H^2-norms are established for this case as well.

In the sixth and concluding chapter, an overview of the results is given. The problem of superconvergence and various modifications of the Galerkin methods for other related problems are discussed. In particular, the Galerkin approximations for one dimensional ablation problem and porous medium equations are investigated.

CHAPTER 2

NONLINEAR STEFAN PROBLEM WITH NEUMANN BOUNDARY CONDITION

2.1 Introduction.

In this chapter we confine our discussion to the singlephase nonlinear Stefan problem with Neumann boundary condition . Earlier, Nitsche [48] considered a linear problem and developed the error analysis for the same. The present work is an extension in the same direction to a class of nonlinear problems. Here we fix the free boundary by a suitable transformation, which results in decoupling the equation into two main systems, leading to considerable simplification in analysis inspite of the presence of the nonlinearity.

In Section 2, we state the basic problem along with regularity conditions. We discuss a co-ordinate transformation to fix the domain in Section 3. Section 4 deals with the weak formulation and an H^1-Galerkin procedure for the transformed problem. In Section 5, the Galerkin approximation is analysed for the local existence and uniqueness. An auxiliary projection is discussed in Section 6 alongwith the related error estimates. The main results are presented in Section 7, where a priori error estimates for the continuous-time Galerkin approximations are established together with the global existence of the approximate solution of the original problem. It is shown that the order of convergence are optimal in H^1 and H^2-norms.

2.2 Statement of the Problem.

We consider the following single phase nonlinear Stefan problem in one space dimension, arising in melting of solid or freezing of liquid [58].

Problem $\tilde{P}$: Find a pair $\{(U,S) : U = U(y,\tau),\ S = S(\tau)\}$ defined on the closure of the parabolic domain

$$\Omega \times (0,T_o],\ \text{where}\ \Omega = \{y : 0 < y < S(\tau)\}\ \text{and}\ 0 < \tau \leq T_o,$$

such that it satisfies

(2.2.1) $$U_\tau - (a(U)U_y)_y = 0,\ (y,\tau)\ \varepsilon\ \Omega \times (0,T_o]$$

and initial, boundary conditions

(2.2.2) $$U(y,0) = g(y),\quad y\ \varepsilon\ I;$$

(2.2.3) $$\left.\begin{aligned} U_y(0,\tau) &= 0, \\ U(S(\tau),\tau) &= 0, \end{aligned}\right. \qquad \tau > 0$$

together with the free boundary relations

(2.2.4) $$S_\tau + a(U)U_y\big|_{y=S(\tau)} = 0,\ \tau > 0$$

and

$$S(0) = 1.$$

Here, we make several assumptions which will be collectively called condition $\tilde{R}_1$.

(2.2.5) Condition $\tilde{R}_1$.

(i) The pair $\{U, S\}$ is a unique solution to (2.2.1) - (2.2.4)

(ii) For $q_1 \varepsilon R$, $0 < \alpha \leq a(q_1)$, where α is a positive constant.

(iii) The function $a(\cdot)$, which depends on U only belongs to $C^2(R)$ with continuous and bounded derivatives. Moreover, there is a constant $K_1 > 0$ such that for $q_1 \varepsilon R$

$$|a(q_1)|,\ \left|\frac{\partial a}{\partial q_1}\right|,\ \left|\frac{\partial^2 a}{\partial q_1^2}\right| \leq K_1.$$

(iv) The initial function g is assumed to be sufficiently smooth and satisfies the compatibility condition,

$$g_y(0) = g(1) = 0.$$

Further, the solution $\{U, S\}$ satisfies the following regularity assumptions :

$$\tilde{R}_2.\quad U \,\varepsilon\, L^\infty(0,T_0;H^{r+1}(\Omega(\tau))) \cap L^\infty(0,T_0;H^2(\Omega(\tau)))$$

$$\cap\, W^{1,2}(0,T_0;H^{r+1}(\Omega(\tau))) \cap W^{1,\infty}(0,T_0;H^1(\Omega(\tau)))$$

and

$$S \,\varepsilon\, W^{1,\infty}(0,T_0).$$

Let $\tilde{K}_2$ denote a bound for the norms of the functions in all of the spaces in $\tilde{R}_2$.

2.3 <u>Straightening the Free Boundary</u>.

In this section, we fix the free boundary using Landau-type transformation [33]

(2.3.1) $$y = x\, S(\tau),\ \tau \geq 0.$$

Further, in order to decouple the resulting system we introduce an additional transformation in time scale, given by

(2.3.2) $$\tau(t) = \int_0^t s^2(t')\, dt'.$$

If we write $U(y,\tau) = u(x,t)$ and $S(\tau) = s(t)$ in the transformed variable, then

(2.3.3) $$U_y(y,\tau) = u_x(x,t)\,\frac{\partial x}{\partial y} = u_x(x,t)\, s^{-1}(t)$$

(2.3.4) $$U_{yy}(y,\tau) = u_{xx}\, s^{-2}(t)$$

and

(2.3.5) $$U_\tau(y,\tau) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial \tau} + u_t\,\frac{\partial t}{\partial \tau}$$

$$= u_t s^{-2}(t) - u_x y s^{-2}(t)\,\frac{ds}{dt}\, s^{-2}(t).$$

But,

(2.3.6) $$S_\tau = \frac{ds}{dt}\frac{dt}{d\tau} = \frac{ds}{dt}\, s^{-2}(t)$$

and from (2.3.3)

(2.3.7) $$U_y(S(\tau),\tau) = u_x(1,t)\, s^{-1}(t).$$

Using (2.3.3) - (2.3.7), the Problem $\tilde{P}$ gets transformed into the Problem P.

<u>Problem</u> P : Find a pair $\{u,s\}$ satisfying

(2.3.8) $$u_t - (a(u)u_x)_x = -a(u(1))u_x(1)x\, u_x,\ (x,t) \in I \times (0,T],$$

with the initial and boundary conditions,

(2.3.9) $\quad u(x,0) = g(x), \quad x \in I;$

(2.3.10) $\quad u_x(0,t) = u(1,t) = 0, \; t > 0$

and

(2.3.11) $\quad \frac{ds}{dt} = - a(u(1)) \, u_x(1)s, \; t > 0$

with $\quad s(0) = 1.$

Clearly, $t = T$ corresponds to $\tau = T_0$. Note that all the regularity assumptions $\tilde{R}_2$ for $\{U,S\}$ easily carry over to $\{u,s\}$ and we call these R_2 with the bound K_2. As $u(1) = 0$, $a(u(1)) = a(0)$, a known constant independent of u. Further, the integral (2.3.2) can be rewritten as

(2.3.12) $\quad \frac{d\tau}{dt} = s^2(t), \; t > 0$

with $\tau(0) = 0.$

Remark. The transformed Problem P involves a nonlinear parabolic problem (2.3.8) – (2.3.10) in a fixed domain and two initial value problems for ordinary differential equations (2.3.11) and (2.3.12). Thus the computation of u can be carried out independent of 's'.

2.4 Weak Formulation and Galerkin Procedure.

Consider the space :

$$\overset{o}{H}{}^2(I) = \{v \in H^2(I) : v_x(0) = v(1) = 0\}.$$

Multiplying both the sides of (2.3.8) by $-v_{xx}$ and integrating by parts the first term with respect to x, we get

$$(2.4.1)\quad (u_{tx}, v_x) + ((a(u)u_x)_x, v_{xx}) = a(0)\, u_x(1)(xu_x, v_{xx}),$$
$$v \in \overset{o}{H}{}^2(I) \text{ and } 0 < t \leq T.$$

If $u \in L^{\infty}(0,T; \overset{o}{H}{}^2(I))$ with $u_t \in L^2(0,T; \overset{o}{H}{}^2(I))$ satisfies (2.4.1) and initial condition $u(x,0) = g(x)$, then it is called a weak solution of (2.3.8) - (2.3.10).

<u>H^1-Galerkin procedure</u>. For each of a family of values of h in $(0,1]$, let $\overset{o}{S}_h \subset \overset{o}{H}{}^2(I)$ be a finite dimensional subspace satisfying both approximation and inverse properties (1.3.6), k = 2 and (1.3.9) respectively. For example of such spaces one can refer to Nitsche [47].

Now we call $u^h : (0,T] \to \overset{o}{S}_h$ an H^1-Galerkin approximation of u, if it satisfies

$$(2.4.2)\quad (u^h_{tx}, \chi_x)+((a(u^h)u^h_x)_x, \chi_{xx}) = a(0)u^h_x(1)(xu^h_x, \chi_{xx}),\ \chi \in \overset{o}{S}_h$$

and the initial condition

$$(2.4.3)\qquad u^h(x,0) = Q_h g(x),$$

where Q_h is an appropriate projection of u onto $\overset{o}{S}_h$ at t = 0, to be defined later.

Further, the Galerkin approximations s_h of s in (2.3.11) and τ_h of τ in (2.3.12) are given by

$$(2.4.4) \qquad \frac{ds_h}{dt} = -a(0)\, u_x^h(1) s_h \ , \text{ with } s_h(0) = 1$$

and

$$(2.4.5) \qquad \frac{d\tau_h}{dt} = s_h^2(t) \ , \text{ with } \tau_h(0) = 0.$$

2.5 Existence and Uniqueness of the Galerkin Approximation.

Now we show the existence and uniqueness of the Galerkin approximation u^h satisfying (2.4.2) – (2.4.3). Since $\overset{o}{S}_h$ is finite dimensional, the equation (2.4.2) results in a system of ordinary differential equations in the temporal variable t in explicit form with differentiable right hand side. Therefore, for a given $u^h(x,0)$ there exists a unique solution to (2.4.2) in a certain interval $(0,t_h)$. We shall now show that t_h does not depend on $\overset{o}{S}_h$, but may depend on the data g. Following Nitsche [48], we have

Theorem 2.5.1. Assume that the projection Q_h is bounded in H^1-norm. Then there is a $\bar{t} > 0$ independent of the choice of the approximation space $\overset{o}{S}_h$ such that u^h is uniquely defined for $t \in (0,\bar{t}]$.

Proof. In order to establish the existence of a $\bar{t}$ independent of $\overset{o}{S}_h$, choose $t \in (0,t_h)$. Setting $\chi = u^h$ in the equation (2.4.2), we get

$$(2.5.1) \quad \frac{1}{2}\frac{d}{dt} \| u_x^h \|^2 + \left(\frac{\partial}{\partial x} (a(u^h)u_x^h), u_{xx}^h\right) = a(0) u_x^h(1) (x u_x^h, u_{xx}^h).$$

But

$$(2.5.2) \quad \left(\frac{\partial}{\partial x} (a(u^h)u_x^h), u_{xx}^h\right) = (a(u^h)u_{xx}^h, u_{xx}^h) + (a_u(u^h)(u_x^h)^2, u_{xx}^h)$$

$$\geq \alpha||u^h_{xx}||^2 - K_1||u^h_x||_{L^\infty}||u^h_x|| \; ||u^h_{xx}||$$

and

(2.5.3) $\quad a(0)u^h_x(1)(xu^h_x, u^h_{xx}) \leq K_1 \; ||u^h_x||_{L^\infty}||u^h_x|| \; ||u^h_{xx}||.$

Since $u^h_x(0) = 0$, then we get by (1.3.15)

(2.5.4) $\quad ||u^h_x||_{L^\infty} \leq \sqrt{2} \; ||u^h_x||^{1/2} \; ||u^h_{xx}||^{1/2}.$

By (2.5.1) - (2.5.4), we now have

$$\frac{d}{dt} ||u^h_x||^2 + 2\alpha||u^h_{xx}||^2 \leq 4\sqrt{2}K_1 \; ||u^h_x||^{3/2} \; ||u^h_{xx}||^{3/2}.$$

Applying Young's inequality (Inequality II of Chapter 1) with a suitable choice of ε and q, where $b = \varepsilon||u^h_{xx}||^{3/2}$ so that $2\alpha||u^h_{xx}||^2 = \frac{b^q}{q}$, we obtain

$$\frac{d}{dt} ||u^h_x||^2 \leq K_3(K_1) \; ||u^h_x||^6.$$

Therefore, $\lambda = ||u^h_x||^2$ obeys the differential inequality

$$\frac{d\lambda}{dt} \leq K_3\lambda^3, \quad \text{for } t > 0$$

with $\qquad \lambda(0) = ||Q_h g_x||^2 \leq K_4||g_x||^2,$

Integrating with respect to t from (0,t), we get

$$-\frac{1}{2} \lambda^{-2}\Big]_0^t \leq K_3 t \implies \lambda^2(t) \leq \frac{\lambda^2(0)}{1-2K_3 t \; \lambda^2(0)}.$$

From which it is clear that $||u^h_x||$ is uniformly bounded in any compact interval of $[0, \frac{1}{2K_5||g_x||^2})$ which is independent of $\overset{\circ}{S}_h$.

Here the constant $K_5 = K_5(K_3, K_4)$. Hence u^h is defined in the above interval.

We have seen earlier that for a given choice of a base, the approximation u^h is chosen uniquely. However, the following uniqueness theorem shows that the approximate solution is independent of the choice of bases.

Theorem 2.5.2 (Uniqueness of the approximate solution). For any $K > 0$, there is atmost one solution $u^h \in \overset{o}{S}_h$ of (2.4.2) – (2.4.3) lying inside a ball B_K, where

$$B_K = \{w(\cdot,t) \in \overset{o}{H}{}^2(I) : \|w\|_{L^\infty(H^2)} \leq K\}.$$

Proof. Let u_1^h and u_2^h both belonging to $\overset{o}{S}_h$ be two solutions of (2.4.2), satisfying the same initial condition. Then the difference $\varphi = u_1^h - u_2^h$ satisfies

$$(2.5.5) \quad (\varphi_{xt}, X_x) + (\frac{\partial}{\partial x}(a(u_1^h)u_{1,x}^h - a(u_2^h)u_{2,x}^h), X_{xx})$$

$$= a(0)u_{1,x}^h(1)(xu_{1,x}^h, X_{xx}) - a(0)u_{2,x}^h(1)(xu_{2,x}^h, X_{xx})$$

and

$$(2.5.6) \qquad \varphi(x,0) = 0.$$

Since $a(u_1^h) - a(u_2^h) = -\int_0^1 \frac{\partial}{\partial u} a(u_1^h - \xi\varphi)\,\varphi\, d\xi$, the equation (2.5.5) can be rewritten as

$$(2.5.7) \quad (\varphi_{xt}, X_x) + (\frac{\partial}{\partial x}(a(u_1^h)\varphi_x), X_{xx}) = (\frac{\partial}{\partial x}(\int_0^1 \frac{\partial a(u_1^h - \xi\varphi)}{\partial u}\varphi\, d\xi\; u_{2,x}^h), X_{xx})$$

$$+ a(0)\{u^h_{1,x}(1)(x\varphi_x, \chi_{xx})+\varphi_x(1)(xu^h_{2,x}, \chi_{xx})\}.$$

Since $\varphi \in \overset{\circ}{S}_h$ we can set $\chi = \varphi$, obtaining

$$(2.5.8)\quad \frac{1}{2}\frac{d}{dt}||\varphi_x||^2+(\frac{\partial}{\partial x}(a(u^h_1)\varphi_x),\varphi_{xx})=(\frac{\partial}{\partial x}(\int_0^1 \frac{\partial a(u^h_1-\xi\varphi)}{\partial u}\varphi d\xi\, u^h_{2,x}),$$

$$\varphi_{xx})+a(0)\{u^h_{1,x}(1)(x\varphi_x, \varphi_{xx})+\varphi_x(1)(xu^h_{2,x}, \varphi_{xx})\} = I_1 + I_2.$$

But as in (2.5.2)

$$(2.5.9)\quad (\frac{\partial}{\partial x}(a(u^h_1)\varphi_x),\varphi_{xx}) \geq \alpha||\varphi_{xx}||^2-K_1||u_{1,x}||_{L^\infty}||\varphi_x||\,||\varphi_{xx}||.$$

Using $||\chi_x||_{L^\infty} \leq ||\chi_{xx}||$ and $||\chi||_{L^\infty} \leq ||\chi_x||$ for $\chi \in \overset{\circ}{S}_h$, it is easily seen that

$$(2.5.10)\quad |I_1| \leq |(\tilde{a}_u\varphi_x u^h_{2,x}, \varphi_{xx})|+|(\tilde{a}_u\varphi u^h_{2,xx}, \varphi_{xx})|$$

$$+ |(\int_0^1 \frac{\partial^2 a(u^h_1-\xi\varphi)}{\partial u^2}(u^h_{1,x}-\xi\varphi_x)\varphi u^h_{2,x}\, d\xi,\ \varphi_{xx})|$$

$$\leq K_1[||u^h_{2,x}||_{L^\infty}||\varphi_x||+||u^h_{2,xx}||\ ||\varphi||_{L^\infty}$$

$$+ ||\varphi_x||\ ||\varphi||_{L^\infty}||u_{2,x}||_{L^\infty}]\ ||\varphi_{xx}||$$

$$\leq K_6(K_1,K)\ ||\varphi_x||\ ||\varphi_{xx}||,$$

where $\tilde{a}_u = \int_0^1 \frac{\partial a(u^h_1-\xi\varphi)}{\partial u} d\xi$.

Further, we have

$$(2.5.11) \quad |I_2| \le |a(0)u^h_{1,x}(1)(x\varphi_x,\varphi_{xx})|+|a(0)\varphi_x(1)(xu^h_{2,x},\varphi_{xx})|$$

$$\le K_1\{|u^h_{1,x}(1)|\ ||\varphi_x||+|\varphi_x(1)|\ ||u^h_{2,x}||_{L^\infty}\}\,||\varphi_{xx}||$$

$$\le K_6(K_1,K)\{||\varphi_x||+|\varphi_x(1,t)|\}\ ||\varphi_{xx}||.$$

From (2.5.8) - (2.5.11), we obtain

$$(2.5.12) \quad \frac{1}{2}\frac{d}{dt}||\varphi_x||^2+\alpha||\varphi_{xx}||^2 \le 3K_6(K_1,K)\{||\varphi_x||\ ||\varphi_{xx}||$$

$$+ |\varphi_x(1)\ ||\varphi_{xx}||\}$$

$$\le 3K_6(K_1,K)\{||\varphi_x||\ ||\varphi_{xx}||$$

$$+ \sqrt{2}\ ||\varphi_x||^{1/2}\ ||\varphi_{xx}||^{3/2}\}.$$

Using the Inequality I of Chapter 1 for the first term on the right hand side of (2.5.12) and Young's inequality (Inequality II, Chapter 1) for the second term with appropriate choices of ε, p and q, we get the terms with $||\varphi_{xx}||^2$ cancelled from both the sides and get

$$(2.5.13) \qquad \frac{d}{dt}||\varphi_x||^2 \le K_7(K_1,K)\ ||\varphi_x||^2.$$

As $\varphi(x,0) = 0$ and consequently $\varphi_x(x,0) = 0$, we have from the inequality (2.5.13)

$$\varphi_x(x,t) \equiv 0.$$

The later shows that $\varphi(x,t)$ is independent of x and vanishes

at $x = 0$, which shows that $\varphi(x,t) \equiv 0$ a.e. This completes the proof of the uniqueness of the approximate solution.

2.6 Some Approximation Lemmas.

Let us now introduce the following forms :

$$(2.6.1) \quad B(u,v;w,\chi) = \frac{1}{2} a(u(1,t))\{v_x(1,t)(xw_x,\chi_{xx}) + w_x(1,t)(xv_x,\chi_{xx})\}$$

and

$$(2.6.2) \quad A(u,v;w) = (\frac{\partial}{\partial x}(a(u)v_x),w_{xx}) - 2B(u,u;v,w).$$

The following Lemma shows the boundedness of A and establishes a Gårding-type inequality for A in $\overset{\circ}{H}{}^2(I)$.

Lemma 2.6.1(i) (Boundedness). $A(u;v,w)$ is bounded in $\overset{\circ}{H}_2(I)$ that is

$$(2.6.3) \quad |A(u;v,w)| \le M\|v_{xx}\| \, \|w_{xx}\|, \text{ for } u \in W^{1,\infty}(I) \text{ and}$$

$v,w \in \overset{\circ}{H}{}^2(I)$, where M may depend on $\|u_x\|_{L^\infty}$.

(ii) (Gårding-type inequality). There exists a constant $\tilde{\alpha} > 0$ and ρ such that

$$(2.6.4) \quad A(u;v,v) \ge \tilde{\alpha}\|v_{xx}\|^2 - \rho\|v_x\|^2, \text{ for } u \in W^{1,\infty}(I) \text{ and}$$

$v \in \overset{\circ}{H}{}^2(I)$. Here ρ may depend on $\|u_x\|_{L^\infty}$.

Proof. From (2.6.2), we have

$$(2.6.5) \quad |A(u;v,w)| \le |(\frac{\partial}{\partial x}(a(u)v_x),w_{xx})| + 2|B(u,u;v,w)|$$

$$\leq |(a(u)v_{xx}, w_{xx})| + |(a_u u_x v_x, w_{xx})| + 2|B(u,u;v,w)|.$$

But, using $|v_x(1)| \leq ||v_{xx}||$, for $v \in \overset{o}{H}{}^2(I)$

(2.6.6) $$2|B(u,u;v,w)| \leq |a(u(1))|\{|u_x(1)(xv_x, w_{xx})|$$

$$+ |v_x(1)(xu_x, w_{xx})|\}$$

$$\leq K_1\{|u_x(1)|\ ||v_x||\ ||w_{xx}|| + |v_x(1)|\ ||u_x||_{L^\infty}||w_{xx}||\}$$

$$\leq K_1||u_x||_{L^\infty}\{||v_x|| + ||v_{xx}||\}||w_{xx}||.$$

Substituting (2.6.6) in (2.6.5), using (2.2.5) condition $\tilde{R}_1$(ii), (iii) as well as the Poincare's inequality (1.3.14) $||v_x|| \leq ||v_{xx}||$, for $v \in \overset{o}{H}{}^2(I)$, we get the estimate (2.6.3).

For the proof of (ii), consider,

(2.6.7) $$A(u;v,v)$$

$$\geq (a(u)v_{xx}, v_{xx}) - \{|(a_u(u)u_x v_x, v_{xx})| + 2|B(u,u;v,v)|\}$$

$$\geq \alpha||v_{xx}||^2 - K_1\{||u_x||_{L^\infty}||v_x|| + ||u_x||_{L^\infty}(|v_x(1)|$$

$$+ ||v_x||)\}||v_{xx}||$$

$$\geq \alpha||v_{xx}||^2 - K_1||u_x||_{L^\infty}\{2||v_x|| + \sqrt{2}||v_x||^{1/2}||v_{xx}||^{\frac{1}{2}}\}||v_{xx}||$$

Apply Young's inequality for the second term in the bracket of right hand side of (2.6.7), we get

(2.6.8) $$\sqrt{2}\,K_1||u_x||_{L^\infty}||v_x||^{1/2}||v_{xx}||^{3/2}$$

$$\leq \tfrac{3}{4}(\delta||v_{xx}||^{3/2})^{4/3} + \tfrac{1}{4}(K_1\,\delta^{-1}||u_x||_{L^\infty}\sqrt{2}||v_x||^{1/2})^4$$

$$= \frac{3}{4}\,\delta^{4/3}\,||v_{xx}||^2 + K_1^4\,\delta^{-4}\,||u_x||^4_{L^\infty}\,||v_x||^2.$$

Further, applying the trivial Inequality I of Chapter 1 for the first term in the bracket of (2.6.7), we get

(2.6.9) $$2K_1||u_x||_{L^\infty}||v_x||\;||v_{xx}|| \le \varepsilon||v_{xx}||^2 + \frac{4K_1^2}{\varepsilon}||u_x||^2_{L^\infty}||v_x||^2.$$

Now choosing $\frac{3}{4}\,\delta^{4/3} = \varepsilon$ and $\varepsilon = \alpha/4$ and applying (2.6.8) – (2.6.9) in (2.6.7), we obtain

$$A(u;v,v) \ge \frac{\alpha}{2}||v_{xx}||^2 - K_1^2||u_x||^2_{L^\infty}\left(\frac{16}{\alpha} + K_1^2||u_x||^2_{L^\infty}\,\frac{27}{\alpha^3}\right)\,||v_x||^2.$$

If we identify $\alpha/2 = \tilde{\alpha}$ and $K_1^2||u_x||^2_{L^\infty}\,\alpha^{-1}(16+27K_1^2||u_x||^2_{L^\infty}\,\alpha^{-2}) = \rho$, then the required estimate (2.6.4) follows.

Let $A_\rho(u;v,w) = A(u;v,w) + \rho(v_x,w_x)$. Then A_ρ is strongly coercive in $\overset{\circ}{H}{}^2(I)$, that is

(2.6.10) $$A_\rho(u;v,v) \ge \tilde{\alpha}\,||v_{xx}||^2, \quad \text{for } u \in W^{1,\infty} \text{ and } v \in \overset{\circ}{H}{}^2.$$

For $t \in [0,T]$, we define $\tilde{u}(x,t) \in \overset{\circ}{S}_h$ as the Galerkin approximation with respect to the form A_ρ :

(2.6.11) $$A_\rho(u;u-\tilde{u},X) = 0, \quad X \in \overset{\circ}{S}_h.$$

The existence and uniqueness of $\tilde{u}$ can be easily proved following the Lax-Milgram Theorem (Theorem 1.3.3), since the bilinear form A_ρ satisfies boundedness and coercive property in $\overset{\circ}{S}_h$ and $F(X) = A_\rho(u;u,X)$ is linear in X and continuous. Here the continuity of $F(X)$ follows from the condition $\tilde{R}_1$.

Below, we derive the estimates for $\eta = u-\tilde{u}$ and its temporal derivative η_t.

Lemma 2.6.2. Let $\eta = u-\tilde{u}$ be defined by (2.6.11). Then, for $t \in [0,T]$, there exists a constant $K_8 = K_8(\alpha,\rho,M,K_0,K_1,K_2)$ such that

(2.6.12) $$||\eta_{xx}|| \leq K_8 h^{m-2} ||\dot{u}||_m \ , \ 2 \leq m \leq r+1.$$

Proof. Take $\chi = \eta - (u-v)$, for $v \in \overset{o}{S}_h$ in (2.6.11). By the coercive property (2.6.10) of A_ρ in $\overset{o}{H}{}^2(I)$, we get

$$\tilde{\alpha}\ ||\eta_{xx}||^2 \leq A_\rho(u;\eta,\eta) = A\ (u;\eta,u-v)$$

$$\leq M\ ||\eta_{xx}|| \inf_{v\in \overset{o}{S}_h} ||u-v||_2.$$

Applying approximation property (1.3.6) for $\overset{o}{S}_h$, we obtain the required estimate (2.6.11).

We shall also require the estimates for the projection error η in L^2 and H^1-norms. In the following, we use K_8 in a generic sense.

Lemma 2.6.3. There exists a constant K_8 as in Lemma 2.6.2 such that

(2.6.13) $$||\eta|| \leq K_8\ h^2\ ||\eta_{xx}||.$$

In particular,

(2.6.14) $$||\eta|| \leq K_8\ h^m\ ||u||_m\ ,\ 2 \leq m \leq r+1.$$

Proof. The proof will follow from the standard Nitsche-Aubin's duality arguments with suitable modifications. Consider one dimensional formal elliptic operator L^* given by

$$L^*(u)w = \frac{\partial}{\partial x}(a(u)w_x) + a(u(1,t))u_x(1,t)(xw)_x - \rho w. \tag{2.6.15}$$

Let $\Psi \in L^2(I)$ and define $\varphi \in H^4(I) \cap \overset{\circ}{H}{}^2(I)$ by

$$L^*(u)\,\varphi_{xx} = \Psi, \quad x \in I; \tag{2.6.16}$$

$$\varphi_{xx}\big|_{x=1} = (xu_x, \varphi_{xx}); \tag{2.6.17}$$

$$\varphi_{xxx}\big|_{x=0} = 0. \tag{2.6.18}$$

Then, for $\eta \in \overset{\circ}{H}{}^2(I)$

$$\begin{aligned}(\eta, L^*\varphi_{xx}) &= (\eta, \tfrac{\partial}{\partial x}(a(u)\varphi_{xxx})) + (\eta, a(u(1))u_x(1,t)(x\varphi_{xx})_x) \\ &\quad - \rho(\eta, \varphi_{xx}) \\ &= -(\eta_x, a(u)\varphi_{xxx}) + \eta a(u)\varphi_{xxx}\Big]_{x=0}^{x=1} \\ &\quad - a(u(1))u_x(1)(x\,\eta_x, \varphi_{xx})\end{aligned} \tag{2.6.19}$$

$$+ a(u(1))u_x(1)x\eta\,\varphi_{xx}\Big]_{x=0}^{x=1} + \rho(\eta_x, \varphi_x) - \rho\eta\varphi_x\Big]_{x=0}^{x=1}$$

(Using integration by parts).

The second, fourth and sixth term vanish because of (2.6.18) and η as well as $\varphi \in \overset{\circ}{H}{}^2(I)$. Integrating by parts, the first term on the right hand side of (2.6.19) with respect to x, and using (2.6.17), we get

$$(2.6.20)\qquad (\eta, L^{*}\varphi_{xx}) = ((a(u)\eta_x)_x, \varphi_{xx}) - a(u)\,\eta_x\varphi_{xx}\Big]_{x=0}^{x=1}$$

$$- a(u(1))u_x(1)\;(x\,\eta_x, \varphi_{xx}) + \rho(\eta_x, \varphi_x)$$

$$= ((a(u)\eta_x)_x, \varphi_{xx}) - a(u(1))\;\eta_x(1)\;\varphi_{xx}(1)$$

$$- a(u(1))u_x(1)\;(x\,\eta_x, \varphi_{xx}) + \rho(\eta_x, \varphi_x)$$

$$= A_\rho(u;\eta, \varphi).$$

Thus defining $D(L^{*})$ as follows :

$$(2.6.21)\quad D(L^{*}) = \{\varphi \in H^4(I) \cap \overset{\circ}{H}{}^2(I) : \varphi_{xx}(1,t) = (xu_x, \varphi_{xx}) \text{ and}$$

$$\varphi_{xxx}(0,t) = 0\},$$

we have a solution $\varphi \in D(L^{*})$, for each $\Psi \in L^2(I)$ satisfying (2.6.16). This follows from the positivity of the operator A_ρ. Here L^{*} is a bounded map on $D(L^{*})$ onto $L^2(I)$ with

$$C_o^{-1}||\Psi|| \leq ||(L^{*})^{-1}\Psi||_2 \leq C_o||\Psi||.$$

Thus, we have

$$(2.6.22)\qquad ||\varphi||_4 \leq C_o||\Psi||,$$

where C_o depends on $||u||_{W^{1,\infty}}$.

Multiplying both the sides of (2.6.16) by η, we get

$$(\eta, \Psi) = (\eta, L^{*}\varphi_{xx}) = A_\rho(u;\eta, \varphi)$$

$$= A_\rho(u;\eta, \varphi - \chi),\ \text{for } \chi \in \overset{\circ}{S}_h$$

$$\leq M\;||\eta_{xx}||\;\inf_{\chi\in\overset{\circ}{S}_h}\;||\varphi - \chi||_2.$$

Here we have used (2.6.20), (2.6.11) and (2.6.3).

By approximation property (1.3.6), k = 2 for $\overset{o}{S}_h$ and the regularity property (2.6.22), we obtain;

$$(2.6.23)\quad (\eta, \Psi) \leq K_o\, M\|\eta_{xx}\|\; h^2\|\varphi\|_4$$

$$\leq MK_oC_oh^2\|\eta_{xx}\|\; \|\Psi\|.$$

Our estimate (2.6.13) for $\|\eta\|$ follows directly from (2.6.23).

To obtain (2.6.14), use the estimate (2.6.12) in (2.6.13).

<u>Lemma</u> 2.6.4. There is a constant K_8 as in Lemma (2.6.2) such that

$$(2.6.24)\qquad \|\eta\|_1 \leq K_8\; h\|\eta_{xx}\|.$$

In particular,

$$(2.6.25)\quad \|\eta\|_1 \leq K_8\; h^{m-1}\; \|u\|_m,\; 2 \leq m \leq r+1.$$

<u>Proof</u>. The result (2.6.24) follows as a consequence of interpolation inequality

$\|\eta\|_1 \leq \|\eta\|^{1/2}\; \|\eta\|_2^{1/2}$, using $\|\eta\| \leq \|\eta_x\|$ and $\|\eta_x\| \leq \|\eta_{xx}\|$, for $\eta \in \overset{o}{H}{}^2(I)$.

Applying the estimate (2.6.12) for $\|\eta_{xx}\|$ in (2.6.24), we get the required estimate for (2.6.25).

In the following Lemma, we obtain an estimate for η_x at x = 1.

<u>Lemma</u> 2.6.5. There exists a constant $K_9 = K_9(\alpha, K_o, K_1, M, K_8)$ such that

(2.6.26) $\quad |\eta_x(1,t)| \leq K_9\, h^{2(m-2)}\, ||u||_m\,,\ 2 \leq m \leq r+1.$

Proof. Define an auxiliary function $\varphi \in H^m(I) \cap \overset{o}{H}{}^2(I)$ satisfying

$$L^*(u)\,\varphi_{xx} = 0,\ x \in I,$$

$$\varphi_{xx}\Big|_{x=1} - (xu_x, \varphi_{xx}) = 1,$$

$$\varphi_{xxx}\Big|_{x=0} = 0.$$

Multiplying both the sides of the first equation by η and integrating by parts, we get

$$\begin{aligned}
|a(u(1))\,\eta_x(1,t)| &= |A_\rho(u;\eta,\varphi)| \\
&= |A_\rho(u;\eta,\varphi-\chi)|,\ \chi \in \overset{o}{S}_h \\
&\leq M||\eta_{xx}|| \inf_{\chi \in \overset{o}{S}_h} ||\varphi-\chi||_2 \\
&\leq MK_0||\eta_{xx}||\, h^{m-2}||\varphi||_m.
\end{aligned}$$

Using the estimate (2.6.12) and $a(u(1)) \geq \alpha$, we get the desired result.

We now wish to find out the bounds for η_t in different norms.

Lemma 2.6.6. For $t \in [0,T]$, there exists a constant $K_{10} = K_{10}(\alpha,\rho,M,K_0,K_1,K_2,K_8,K_9)$ such that

(2.6.27) $\quad ||\eta_{txx}|| \leq K_{11}\, h^{m-2}(||u_t||_m + ||u||_m),\ 2 \leq m \leq r+1.$

Proof. Differentiating (2.6.11) with respect to 't', we get

(2.6.28) $A_\rho(u;\eta_t,v) = -(\frac{\partial}{\partial x}(\eta_x \frac{d}{dt}(a(u))), v_{xx}) + 2B(u,u_t;\eta,v).$

Let $u^* \varepsilon \overset{o}{S}_h$ satisfy

(2.6.29) $$A_\rho(u;u_t-u^*,v) = 0, \quad v \varepsilon \overset{o}{S}_h.$$

Choosing $v = (u_t-u^*)-(u_t-\chi)$ in (2.6.29), for $\chi \varepsilon \overset{o}{S}_h$ and applying boundedness and coercive property for A_ρ, we get

$$\tilde{\alpha}||(u_t-u^*)_{xx}||^2 \leq A_\rho(u;u_t-u^*,u_t-u^*)$$

$$= A_\rho(u;u_t-u^*, u_t-\chi)$$

$$\leq M||(u_t-u^*)_{xx}|| \inf_{\chi\varepsilon\overset{o}{S}_h} ||u_t-\chi||_2.$$

Therefore, by the approximation property (1.3.6), k = 2 for the space $\overset{o}{S}_h$

(2.6.30) $$||(u_t-u^*)_{xx}|| \leq \tilde{\alpha}^{-1} MK_0||u_t||_m h^{m-2}, \quad 2 \leq m \leq r+1.$$

Subtracting (2.6.29) from (2.6.28) and choosing $v = u^*-\tilde{u}_t$, we obtain

$$\tilde{\alpha}||u^*-\tilde{u}_t||^2 \leq A_\rho(u;u^*-\tilde{u}_t,u^*-\tilde{u}_t)$$

$$\leq |(\frac{\partial}{\partial x}(a(u)_t \eta_x),(u^*-\tilde{u}_t)_{xx})|+2|B(u,u_t;u^*-\tilde{u}_t)|$$

$$\leq K_1||u_{tx}||_{L^\infty}(||\eta_{xx}||+||\eta_x||+|\eta_x(1,t)|)||(u^*-\tilde{u}_t)_{xx}||.$$

Hence, from (2.6.12), (2.6.24) and (2.6.26)

(2.6.31) $$||u^*-\tilde{u}_t|| \leq \tilde{\alpha}^{-1}K_1K_2(K_8||u||_m h^{m-2}+K_8||u||_m h^{m-1} + K_9||u||_m h^{2(m-2)}).$$

From (2.6.30) and (2.6.31), the result follows.

The following Lemma gives the estimates of η_t in L^2-norm.

We use in the following K_{10} as a generic constant.

<u>Lemma</u> 2.6.7. There is a constant K_{10} as in Lemma 2.6.6 such that

$$(2.6.32) \qquad ||\eta_t|| \le K_{10}\, h^m(||u_t||_m + ||u||_m), \text{ for } 4 \le m \le r+1.$$

<u>Proof</u>. The proof makes use of duality argument in a modified form.

Let $\varphi \in D(L^*)$ be the solution of

$$L^* \varphi_{xx} = \Psi \ , \text{ for } \Psi \in L^2(I).$$

Clearly, by regularity property (2.6.22) $||\varphi||_4 \le C_o||\Psi||$.

Multiplying both the sides by η_t, we get as in (2.6.28)

$$(2.6.33) \quad (\eta_t, \Psi) = (\eta_t, L^* \varphi_{xx})$$

$$= A_\rho(u; \eta_t, \varphi - \chi) - (\frac{\partial}{\partial x}(a(u)_t\ \eta_x),\ \chi_{xx})$$

$$+ 2B(u, u_t; \eta, \chi)$$

$$= A_\rho(u; \eta_t, \varphi-\chi) + (\frac{\partial}{\partial x}(a(u)_t\ \eta_x), (\varphi-\chi)_{xx})$$

$$- 2B(u, u_t; \eta, \varphi - \chi) - (\frac{\partial}{\partial x}(a(u)_t\ \eta_x),\ \varphi_{xx}) + 2B(u, u_t; \eta, \varphi)$$

$$= I_1 + I_2 + I_3 + I_4 + I_5.$$

The I_2 and I_3 terms are bounded by

$$(2.6.34) \qquad K_1K_2(||\eta_x|| + |\eta_x(1,t)|) \inf_{\chi \in \overset{o}{S}_h} ||\varphi - \chi||_2,$$

(2.6.35) $$|I_1| \leq K_1K_2||\eta_{txx}|| \inf_{\chi\in \overset{o}{S}_h} ||\varphi-\chi||_2.$$

For I_4, integrating by parts we get

$$I_4 = (\eta_x, a(u)_t \ \varphi_{xxx}) = -(\eta, \frac{\partial}{\partial x}(a(u)_t \ \varphi_{xxx})), \text{for } \eta \in \overset{o}{H}{}^2(I)$$

and $\varphi \in D(L^*)$.

Therefore,

(2.6.36) $$|I_4| \leq K_1K_2(||\varphi_{xxx}|| + ||\varphi_{xxxx}||) \ ||\eta||$$
$$\leq K_1K_2 \ ||\varphi||_4 \ ||\eta||.$$

Analogously, we obtain

(2.6.37) $$|I_5| \leq K_1K_2(||\eta|| + |\eta_x(1,t)|) \ ||\varphi||_4.$$

Combining (2.6.33) - (2.6.37) and applying the regularity property (2.6.22) and the estimates in (2.6.14) and (2.6.25) - (2.6.27), we get

$$(\eta_t, \Psi) \leq K_oK_1K_2 \ h^2[K_8||u||_m h^{m-1} + K_9||u||_m h^{2(m-2)} + K_8||u||_m h^{m-2} + K_{10}h^{m-2}(||u_t||_m + ||u||_m) \quad ||\varphi||_4 + K_1K_2(K_8||u||_m h^m + K_9||u||_m h^{2(m-2)}) \ ||\varphi||_4$$
$$\leq C_oK_oK_1K_2(K_8||u||_m h^m + K_{10}(||u||_m + ||u_t||_m)h^m + K_9||u||_m h^{2(m-2)})||\Psi||$$

Therefore, for $2m-4 \geq m$ that is $r+1 \geq m \geq 4$

(2.6.38) $$(\eta_t, \Psi) \leq K_{10} \ h^m(||u||_m + ||u_t||_m) \ ||\Psi||.$$

and the result follows directly from (2.6.38).

Remark. The estimate for $||\eta_{tx}||$ can be proved for $m \geq 3$ that is $r \geq 2$, if we consider $\Psi = \eta_{txx}$ in the proof of the Lemma 2.6.7 or by interpolation similar to the Lemma 2.6.4 between $||\eta_{txx}||$ and $||\eta_t||$.

2.7 A Priori Error Estimates for Continuous Time Galerkin Approximation.

In this Section, we consider the global existence of u^h and a priori estimates for $u-u^h$. Denote $e = u-u^h$, $\eta = u-\tilde{u}$ and $\zeta = u^h-\tilde{u}$, then $e = \eta-\zeta$. Let $u^h(x,0) = Q_h g$ be defined as the Galerkin solution of $u(x,0)$ with respect to the form A_ρ :

$$(2.7.1) \quad A_\rho(u;u-u^h,X) = 0 \text{ at } t = 0, \; X \varepsilon \overset{o}{S}_h \text{ i.e. } A_\rho(g,g-Q_h g,X) = 0.$$

Since g is sufficiently smooth, the projection $Q_h g$ is defined uniquely by Lax-Milgram Theorem (Theorem 1.3.3). Clearly, $u^h(x,0) \equiv \tilde{u}(x,0)$.

Below, we rewrite the Galerkin approximation equation in a suitable form. For this purpose, we have from equation (2.4.1) and (2.6.11)

$$(2.7.2) \quad (\tilde{u}_{tx},X_x)+A_\rho(u;\tilde{u},X) = -(\eta_{tx},X_x)-B(u,u;u,X)+\rho(u_x,X_x), \; X\varepsilon \overset{o}{S}_h.$$

Subtracting (2.7.2) from (2.4.2) and setting $\zeta = u^h-\tilde{u}$, we get

$$(2.7.3) \quad (\zeta_{tx},X_x)+A_\rho(u^h;u^h,X)-A_\rho(u;\tilde{u},X) \equiv (\eta_{tx},X_x)-B(u^h,u^h;u^h,X)$$

$$+ B(u,u;u,X)+ \rho(u^h_x-u_x,X_x), \; X\varepsilon \overset{o}{S}_h.$$

But,

$$(2.7.4)\quad A_\rho(u^h;u^h,\chi) - A_\rho(u;\tilde{u},\chi) = A_\rho(u;\zeta,\chi)+(\frac{\partial}{\partial x}([a(u^h)-a(u)]\ u_x^h),$$
$$\chi_{xx})+2B(u,u;u^h,\chi)-2B(u^h,u^h;u^h,\chi).$$

Further,

$$(2.7.5)\quad \frac{\partial}{\partial x}\ ([a(u^h)-a(u)]\ u_x^h) = -\frac{\partial}{\partial x}\ (\tilde{a}_u\ e\ u_x^h)$$
$$= -\int_0^1 \frac{\partial^2 a(u-\xi e)}{\partial u^2}(u_x-\xi e_x)\ d\xi\ e\ u_x^h$$
$$+\ \tilde{a}_u\ e_x\ u_x^h + \tilde{a}_u\ e\ u_{xx}^h\ ,$$

where $\quad \tilde{a}_u = \int_0^1 \frac{\partial a(u-\xi e)}{\partial u}\ d\xi.$

Substituting the values (2.7.4) - (2.7.5) in (2.7.3), we obtain

$$(2.7.6)\quad (\zeta_{tx},\chi_x)+A_\rho(u;\zeta,\chi) = (\eta_{tx},\chi_x)+B(u,u;u,\chi)-B(u^h,u^h;u^h,\chi)$$
$$-\ \rho(\eta_x,\chi_x) + \rho(\zeta_x,\chi_x)+(\tilde{a}_u(e_xu_x^h+eu_{xx}^h),\chi_{xx})$$
$$+\ (\int_0^1 \frac{\partial^2 a(u-\xi e)}{\partial u^2}(u_x-\xi e_x)\ d\xi\ e\ u_x^h,\chi_{xx})$$
$$-\ 2B(u,u;u^h,\chi) + 2B(u^h,u^h;u^h,\chi).$$

Now, for u and $u^h \in \overset{o}{H}{}^2$ we have $a(u(1)) = a(u^h(1)) = a(0)$ and

$$(2.7.7)\quad B(u,u;u,\chi) - 2B(u,u;u^h,\chi) + B(u^h,u^h;u^h,\chi)$$
$$= B(u,u;e,\chi) - B(u,u^h;e,\chi)$$
$$= B(u,e;e,\chi).$$

From (2.7.6) and (2.7.7), we get

$$(2.7.8)\quad (\zeta_{tx}, \chi_x) + A_\rho(u; \zeta, \chi) = (\eta_{tx}, \chi_x) + \rho(\zeta_x, \chi_x) - \rho(\eta_x, \chi_x)$$
$$+ a(0)e_x(1,t)(x\,\eta_x, \chi_{xx}) - a(0)e_x(1,t)\,(x\zeta_x, \chi_{xx})$$
$$+ \left(\int_0^1 \frac{\partial a}{\partial u}(u-\xi e)\,d\xi\ (\eta_x-\zeta_x)(e_x+u_x),\ \chi_{xx}\right)$$
$$+ \left(\int_0^1 \frac{\partial a}{\partial u}(u-\xi e)\ d\xi\ (\eta-\zeta)(e_{xx}+u_{xx}),\ \chi_{xx}\right)$$
$$+ \left(\int_0^1 \frac{\partial^2 a}{\partial u^2}(u-\xi e)\ d\xi\ u_x(\eta-\zeta)(e_x+u_x),\ \chi_{xx}\right)$$
$$- \left(\int_0^1 \frac{\partial^2 a}{\partial u^2}(u-\xi e)\,\xi\ d\xi\ (\eta_x-\zeta_x)e\ (e_x+u_x),\ \chi_{xx}\right).$$

This equation can be interpreted as an implicit equation for u^h given in terms of ζ and e. Integrating by parts the first and third terms in the right hand side of (2.7.8) and replacing e by E, for some function $E(x,t) \in \overset{o}{H}{}^2(I)$ for each t, we obtain

$$(2.7.9)\quad (\zeta_{tx}, \chi_x) + A_\rho(u; \zeta, \chi) = -(\eta_t, \chi_{xx}) + \rho(\zeta_x, \chi_x) + \rho(\eta, \chi_{xx})$$
$$+ a(0)E_x(1,t)(x\,\eta_x, \chi_{xx}) - a(0)E_x(1,t)(x\zeta_x, \chi_{xx})$$
$$+ \left(\int_0^1 \frac{\partial a}{\partial u}(u-\xi E)\ d\xi\ (\eta_x-\zeta_x)(E_x+u_x),\ \chi_{xx}\right)$$
$$+ \left(\int_0^1 \frac{\partial a}{\partial u}(u-\xi E)\ d\xi\ (\eta-\zeta)\ (E_{xx}+u_x),\ \chi_{xx}\right)$$
$$+ \left(\int_0^1 \frac{\partial^2 a}{\partial u^2}(u-\xi E)\ d\xi\ u_x(\eta-\zeta)(E_x+u_x),\ \chi_{xx}\right)$$
$$- \left(\int_0^1 \frac{\partial^2 a}{\partial u^2}(u-\xi E)\ \xi\ d\xi\ (\eta_x-\zeta_x)E(E_x+u_x),\ \chi_{xx}\right).$$

This is a linear equation in ζ for each E with

(2.7.10) $\qquad \zeta(x,0) = 0.$

Hence, there exists a unique solution ζ of (2.7.9)-(2.7.10) for any given $E \in \overset{\circ}{H}{}^2(I)$. Thus, we have defined by this process an operator $\zeta = \mathcal{J}E$ on $\overset{\circ}{H}{}^2(I)$ through the equation (2.7.9) and (2.7.10). Clearly $\mathcal{J}E = \eta - e$.

Now, the problem of existence of u^h reduces to finding a fixed point for the operator $e = \eta - \mathcal{J}E$. In otherwords, since $e = \eta - \zeta$ and ζ depends on E, e also depends on E. We show that there is an E such that e = E.

Lemma 2.7.1. For a given $E(x,t) \in \overset{\circ}{H}{}^2(I)$, we have the following estimate for ζ in (2.7.9)

(2.7.11) $$\|\zeta_x\|_{L^\infty(L^2(I))}$$

$$\leq K_{11}\{\|\eta_t\|_{L^2(L^2)} + (\|E\|_{L^\infty(H^2)}(1+\|E\|_{L^\infty(H^2)}) + 1)$$

$$\|\eta_x\|_{L^1(H^2)} + \|\eta\|\} \quad \exp\,[TK_{12}\{\|E\|_{L^\infty(H^2)}$$

$$(1+\|E\|_{L^\infty(H^2)}) + \rho\}],$$

where K_{11} are constants depending on K_1 and K_2.

Proof. Setting $X = \zeta$ in (2.7.9) and using (2.2.5) condition $\tilde{R}_1$ and R_2 we obtain

$$\frac{1}{2}\frac{d}{dt}\|\zeta_x\|^2+\tilde{\alpha}\|\zeta_{xx}\|^2 \le \|\eta_t\|\,\|\zeta_{xx}\|+\rho\|\zeta_x\|^2+\rho\|\eta\|\,\|\zeta_{xx}\|$$

$$+K_1|E_x(1)|\,\|\eta_x\|\,\|\zeta_{xx}\|+K_1|E_x(1)|\,\|\zeta_x\|\,\|\zeta_{xx}\|$$

$$+K_1(\|\eta_x\|+\|\zeta_x\|)\|E_x\|_{L^\infty}\|\zeta_{xx}\|+K_1K_2(\|\eta_x\|+\|\zeta_x\|)\|\zeta_{xx}\|$$

$$+K_1(\|\eta\|_{L^\infty}+\|\zeta\|_{L^\infty})(\|E_{xx}\|+K_2)\|\zeta_{xx}\|+K_1K_2(\|\eta\|+\|\zeta\|)$$

$$(\|E_x\|_{L^\infty}+K_2)\|\zeta_{xx}\|+K_1(\|\eta_x\|+\|\zeta_x\|)\|E\|_{L^\infty}(\|E_x\|_{L^\infty}+K_2)\|\zeta_{xx}\|.$$

Noting $\|E_x\|_{L^\infty} \le \|E\|_{H^2}$, for $E \in \overset{\circ}{H}{}^2(I)$ and using $ab \le \frac{1}{\varepsilon}a^2+\varepsilon b^2$, $a,b \ge 0$, $\varepsilon > 0$ to the right hand expression with $b = \|\zeta_{xx}\|$, we get

$$(2.7.12)\quad \frac{d}{dt}\|\zeta_x\|^2+2\tilde{\alpha}\|\zeta_{xx}\|^2$$

$$\le 40\varepsilon\|\zeta_{xx}\|^2+\frac{2}{\varepsilon}\|\eta_t\|^2+K_{12}(K_1,K_2,\varepsilon)\,(\|E\|_2^2$$

$$+\|E\|_2^2(1+\|E\|_2^2)+1)\,\|\eta_x\|^2+K_{13}(K_1,K_2,\rho,\varepsilon)\|\eta\|^2$$

$$+K_{12}(K_1,K_2,\varepsilon)\,(\|E\|_2^2(1+\|E\|_2^2)+\rho)\,\|\zeta_x\|^2.$$

Here we have used Propositions 1.3.5 and 1.3.8 to estimate $\|\varphi\|$ and $\|\varphi\|_{L^\infty}$ by $\|\varphi_x\|$, since $\varphi \in \overset{\circ}{H}{}^2(I)$.

Choose ε so that the term $\|\zeta_{xx}\|$ cancels from both sides of the equation (2.7.12). Now, applying Gronwall's Lemma (Lemma 1.3.9) and taking supremum over the interval $[0,T]$, we get the required estimate (2.7.11).

The following theorem shows the existence of a unique solution satisfying $e(E) = E$. Further, an order of magnitude is obtained for the same.

Theorem 2.7.2. Suppose u satisfies the regularity condition R_2. Then, for sufficiently small h there exists exactly one solution u^h in the neighbourhood of u such that

$$(2.7.13)\quad ||e||_{L^\infty(H^2)} \leq K_{14}\, h^{m-2},\ 4 \leq m \leq r+1,$$

where, K_{14} is a constant depending on K_o, K_8, K_{10}, K_{11} and K_{12}.

Proof. In the estimate (2.7.11), using (2.6.14), (2.6.25) and (2.6.32), we get for $m \geq 4$

$$(2.7.14)\quad ||\zeta_x||_{L^\infty(L^2)} \leq K_{11}\{K_{10}(||u_t||_{L^2(H^m)}+||u||_{L^2(H^m)})h^m$$

$$+ K_8||u||_{L^1(H^m)}h^{m-1}(||E||_{L^\infty(H^2)}(1+||E||_{L^\infty(H^2)})+1)$$

$$+ K_8||u||_{L^2(H^m)}h^m\}\exp[K_{12}T\{||E||_{L^\infty(H^2)}(1+||E||_{L^\infty(H^2)})+\rho\}].$$

Now, using the inverse property (1.3.9), for the space $\overset{o}{S}_h$ and $||\zeta_x|| \leq ||\zeta_{xx}||$, as $\zeta \in \overset{o}{S}_h$ we have

$$||e||_{L^\infty(H^2)} \leq ||\eta||_{L^\infty(H^2)} + ||\zeta||_{L^\infty(H^2)}$$

$$\leq K_8||u||_{L^\infty(H^m)}h^{m-2}+K_o h^{-1}||\zeta||_{L^\infty(H^1)}$$

$$\leq K_{15}(K_o,K_8,K_{10},K_{11})h^{m-2}\ [||u||_{L^\infty(H^m)}$$

$$+ \{||u||_{L^2(H^m)}+||u_t||_{L^2(H^m)}+||u||_{L^2(H^m)}(||E||_{L^\infty(H^2)}$$

$$(1+||E||_{L^\infty(H^2)})+ 1)\}]\exp [K_{12}T\{||E||_{L^\infty(H^2)}(1+||E||_{L^\infty(H^2)})+\rho\}].$$

In the ball $||E||_{L^\infty(H^2)} \leq 1$, we have

$$(2.7.15) \quad ||e||_{L^\infty(H^2)} \leq K_{15}h^{m-2} [||u||_{L^\infty(H^m)}+\{||u||_{L^2(H^m)}$$

$$+ ||u_t||_{L^2(H^m)}\} \exp \{K_{12}T(2+\rho)\}].$$

For h sufficiently small,

$$||e||_{L^\infty(H^2)} \leq 1.$$

Thus, the map $E \rightarrow e$ is mapping the unit sphere into itself, for sufficiently small h. But it is a continuous map on a subset of a finite dimensional space and hence compact. So, by Schauder's fixed point theorem (Theorem 1.3.11), there exists an E such that $e(E) = E$. Further,

$$||e||_{L^\infty(H^2)} \leq K_{14}\, h^{m-2},\ 4 \leq m \leq r+1,$$

where K_{14} depends on K_o, K_2, K_8, K_{10}, K_{11} and K_{12}.

<u>Corollary</u> 2.7.3. The following estimate holds for e

$$(2.7.16) \quad ||e||_{L^\infty(H^1)} \leq K_{16}\, h^{m-1},\ 4 \leq m \leq r+1,$$

where K_{16} depends on K_{15}, K_8 and K_2.

<u>Proof</u>. Replacing E by e in (2.7.14) and using estimate (2.7.13), we obtain

$$(2.7.17) \qquad \|\zeta_x\|_{L^\infty(L^2)} \leq K_{15}\, h^{m-1}.$$

Further by the inequality (1.3.14),

$$\|\zeta\| \leq \|\zeta_x\|,$$

and from (2.7.17), we have

$$\|\zeta\|_{L^\infty(H^1)} \leq K(K_{15})\, h^{m-1}.$$

Besides, we get from (2.6.25)

$$\|e\|_{L^\infty(H^1)} \leq \|\eta\|_{L^\infty(H^1)} + \|\zeta\|_{L^\infty(H^1)}$$

$$\leq K_8\, h^{m-1}\, \|u\|_{L^\infty(H^m)} + K(K_{15})\, h^{m-1}.$$

Therefore, the result follows.

We are now looking for a pair of approximate solutions of $\{U,S\}$, where $\{U,S\}$ is the solution of the original problem (2.2.1) - (2.2.4). The Galerkin approximations U^h and S_h are given by

$$(2.7.18) \qquad U^h(y,\tau) = u^h(x,t), \qquad S_h(\tau) = s_h(t),$$

where

$$(2.7.19) \qquad y = s_h(t)x, \qquad \tau = \tau_h(t)$$

and s_h, τ_h are given by the equations (2.4.4), (2.4.5) respectively.

The conclusive theorem concerning the error estimates in $U-U^h$, $S-S_h$ and $\tau - \tau_h$ is given as follows :

Theorem 2.7.4. Under the assumptions of Theorem 2.7.3, (2.2.5) condition $\tilde{R}_1$ and $\tilde{R}_2$, the following error estimates hold for $r \geq 3$

(2.7.20) $$||S-S_h||_{L^\infty(0,T_0)} = O(h^r),$$

(2.7.21) $$||\tau-\tau_h||_{L^\infty(0,T_0)} = O(h^r)$$

and

(2.7.22) $$|||U-U^h|||_{L^\infty(0,T_0;H^j(\tilde{\Omega}(\tau)))} = O(h^{r+1-j}), \quad j = 1,2,$$

where the last norm is understood in the following sense

(2.7.23) $$|||\varphi|||_{L^\infty(0,T_0;H^j(\tilde{\Omega}(\tau)))} = \sup_{0<\tau\leq T_0} \{||\varphi||_{H^j(\tilde{\Omega}(\tau))}\},$$

with

(2.7.24) $$\tilde{\Omega}(\tau) = (0, \min (S(\tau), S_h(\tau))).$$

Proof. Let $e_1 = s-s_h$. From (2.3.11) and (2.4.4), we get

(2.7.25) $$\frac{de_1}{dt} = -a(0)\,[e_x(1,t)s + u_x^h(1)e_1].$$

Integrating (2.7.25) with respect to t, we have

$$|e_1(t')| = |-a(0) \int_0^{t'} [e_x(1,t)s(t)+u_x^h(1)e_1(t)]\,dt|$$
$$\leq K_1 \int_0^{t'} (|e_x(1,t)|\,|s(t)|+|u_x^h(1)|\,|e_1(t)|)\,dt.$$

Since,

$$|u_x^h(1,t)| \le ||u_{xx}^h||$$

$$\le ||e_{xx}|| + ||u_{xx}||$$

$$\le K_{14}h^{r-1} + K_2 \le 2K_2, \text{ for sufficiently small } h \text{ and } r \ge 3,$$

we apply Gronwall's Lemma (Lemma 1.3.9) to get

(2.7.26) $$||e_1||_{L^\infty(0,T)} \le K(K_1, K_2)(||\eta_x(1,\cdot)||_{L^2(0,T)} + ||\zeta_x(1,\cdot)||_{L^2(0,T)}) \exp(2K_2K_1T).$$

But, from (1.3.18) we have

(2.7.27) $$||\zeta_x(1,\cdot)||_{L^2(0,T)} \le ||\zeta_{xx}||_{L^2(0,T;L^2)} .$$

For the estimate $||\zeta_{xx}||_{L^2(0,T;L^2)}$, choose ε in the inequality (2.7.12) so that $2\tilde{\alpha}-1 = 40\varepsilon$ and replacing E by e, we get by (2.6.14), (2.6.25), (2.6.32) and (2.7.13)

(2.7.28) $$||\zeta_{xx}||_{L^2(0,T;L^2)} \le K_{15}h^r.$$

From (2.7.26) - (2.7.28) and (2.6.26), we obtain as estimate

(2.7.29) $$||e_1||_{L^\infty(0,T)} = O(h^r), \ r \ge 3.$$

The estimate (2.7.20) follows, if we note

$$||S-S_h||_{L^\infty(0,T_0)} = ||s-s_h||_{L^\infty(0,T)} .$$

Subtracting (2.4.5) from (2.3.12) and then integrating with respect to t, we have

$$\tau(t) - \tau_h(t) = \int_0^t (s^2 - s_h^2)\, dt' .$$

Now from (2.7.29), the estimate (2.7.21) can be easily derived. Finally, we have

$$|||U-U^h|||_{L^\infty(0,T_0;H^j(\tilde{\Omega}(\tau)))} \leq ||u-u^h||_{L^\infty(0,T;H^j)} .$$

Hence, using (2.7.16) and (2.7.13) the estimate (2.7.22), for $j = 1,2$ follows.

CHAPTER 3

NONLINEAR STEFAN PROBLEM WITH DIRICHLET BOUNDARY CONDITION

3.1 Introduction.

In this chapter, we consider a single phase nonlinear Stefan problem with Dirichlet boundary conditions. Based on straightening the free boundary by a co-ordinate transformation, an H^1-Galerkin procedure is analysed both for continuous as well as discrete time Galerkin approximations. The global existence of the Galerkin approximation is established and a priori estimates are obtained.

The organization of the chapter is as follows : In Section 2, the basic problem and its transformed system are discussed. The Section 3 deals with the weak formulation and H^1-Galerkin procedures. In Section 4, an auxiliary projection is defined and its error estimates are derived. Assuming existence and uniqueness of the Galerkin approximations, a priori error estimates for the continuous time Galerkin approximations are obtained in Section 5. In Section 6, the global existence and uniqueness of the approximate solution are established. Finally in Section 7, a priori error estimates for discrete time Galerkin approximations are derived.

3.2 Problem and Co-ordinate Transformation.

We consider the following single phase nonlinear Stefan problem in one space dimension :

Find a pair $\{U, S\}$ so that $U = U(y,\tau)$ satisfies the nonlinear parabolic equation

(3.2.1) $$U_\tau - (a(U)U_y)_y = 0 \text{ in } \Omega \times (0,T_o] ,$$

where $\Omega(\tau) = \{y : 0 < y < S(\tau)\}$ and $0 < \tau \leq T_o$ with the initial and boundary conditions

(3.2.2) $$U(y,0) = g(y), \quad y \in I ;$$

(3.2.3) $$U(0,\tau) = 0 \quad , \quad 0 < \tau \leq T_o ,$$
$$U(S(\tau),\tau) = 0, \quad 0 < \tau \leq T_o,$$

where the free boundary $S = S(\tau)$ satisfies an additional condition of the form :

(3.2.4) $$S_\tau + a(U)U_y\Big|_{y=S(\tau)} = 0, \quad 0 < \tau \leq T_o$$

with $S(0) = 1$.

Here, we make several assumptions, which will be collectively called Condition B

(3.2.5) CONDITION B.

(i) The pair $\{U, S\}$ is a unique solution to (3.2.1)-(3.2.4).

(ii) For all $p \in R$, $0 < \alpha \leq a(p)$, where α is a positive constant.

(iii) The function $a(\cdot)$, which depends on U only belongs to $C^2(R)$ with continuous and bounded derivatives. Moreover, there is a constant $K_1 > 0$ such that

$$|a|, \ |a_p|, \ |a_{pp}| \leq K_1.$$

(iv) The initial function g is sufficiently smooth and

$g(0) = g(1) = 0.$

We organize our regularity hypothesis on the solution $\{U,S\}$ of (3.2.1) - (3.2.4) according to the results in which they are used. We assume

$\tilde{R}_1.$ $U \in L^\infty(0,T_0;H^{r+1}(\Omega(\tau))) \cap L^\infty(0,T_0;H^2(\Omega(\tau)))$

$\cap W^{1,2}(0,T_0;H^{r+1}(\Omega(\tau))) \cap W^{1,\infty}(0,T_0;H^1(\Omega(\tau))),$

$S \in W^{1,\infty}(0,T_0).$

$\tilde{R}_2.$ $U \in \tilde{R}_1 \cap W^{2,2}(0,T_0;H^2(\Omega(\tau))) \cap W^{1,2}(0,T_0;W^{1,\infty}(\Omega(\tau))),$

$S \in \tilde{R}_1 \cap W^{2,2}(0,T_0).$

Let $\tilde{K}_2$ denote a bound for U and S in all of the spaces described in $\tilde{R}_1$ and $\tilde{R}_2$.

Below, we fix the free boundary using Landau type transformation

$$x = S^{-1}(\tau)y. \tag{3.2.6}$$

Further, in order to decouple the resulting system, we introduce an additional transformation in time scale given by

$$t = t(\tau) = \int_0^\tau S^{-2}(\tau')\, d\tau' . \tag{3.2.7}$$

An easy calculation shows as in 2.3 that the function $u(x,t) = U(y,\tau)$ satisfies

(3.2.8) $u_t-(a(u)u_x)_x=-a(u(1))u_x(1)xu_x, (x,t) \in I \times (0,T]$,

with the initial and boundary conditions,

(3.2.9) $u(x,0) = g(x) \; , \; x \in I;$

(3.2.10) $u(0,t) = u(1,t) = 0, \; t > 0$

and the function $s(t) = S(\tau)$ satisfies

(3.2.11) $s_t + a(u(1)) \; u_x(1)s = 0, \quad t > 0$

with $s(0) = 1.$

Here, $t = T$ corresponds to $\tau = T_o$. Note that all the smoothness assumptions $\tilde{R}_1$ and $\tilde{R}_2$ on the pair $\{U,S\}$ carry over to $\{u,s\}$ with the bound K_2 (say). For convenience, let us call these regularity conditions on $\{u,s\}$ R_1 and R_2. As $u(1) = 0$ $a(u(1)) = a(0)$, a known constant independent of u.

Further, the integral (3.2.7) can be rewritten as

(3.2.12) $\frac{d\tau}{dt} = s^2(t), \quad t > 0$

with $\tau(0) = 0.$

3.3 Weak Formulation and Galerkin Procedures.

Let $H^2(I) \cap H_o^1 = \{u \in H^2(I) \; : \; u(0) = u(1) = 0\}.$ Multiplying both the sides of (3.2.8) by $-v_{xx}$ and integrating by parts the first term with respect to x, we get

(3.3.1) $(u_{tx}, v_x)+((a(u)u_x)_x, v_{xx})=a(0)u_x(1)(xu_x, v_{xx}), v \in H^2(I).$

If $u \in L^{\infty}(0,T;H^2(I) \cap H_0^1(I))$ satisfies (3.3.1) and initial condition (3.2.9), then it is called a weak solution of (3.2.8) – (3.2.10).

<u>H^1-Galerkin procedure</u>. Let $\overset{\circ}{S}_h \subset H^2 \cap H_0^1$ be a finite dimensional subspace, satisfying the approximation property (1.3.6), for $k = 2$ and inverse property (1.3.9). Then, the Galerkin procedure is defined as follows : Find $u^h : (0,T] \to \overset{\circ}{S}_h$ such that

$$(3.3.2)\quad (u^h_{tx}, \chi_x)+((a(u^h)u^h_x)_x, \chi_{xx})=a(0)u^h_x(1)(xu^h_x, \chi_{xx}),\ \chi \in \overset{\circ}{S}_h$$

and $$u^h(x,0) = Q_h g(x),$$

where Q_h is an appropriate projection to be defined later. Moreover, the Galerkin approximations s_h and τ_h of s and τ respectively are defined by

$$(3.3.3)\qquad \frac{ds_h}{dt} = -a(0)\ u^h_x(1)\ s_h,\quad t > 0$$

with $s_h(0) = 1$

and

$$(3.3.4)\qquad \frac{d\tau_h}{dt} = s_h^2(t),\ t > 0$$

with $\tau_h(0) = 0.$

Next, we define a fully discrete, $O(\Delta t)$ – correct method for approximating $\{u,s\}$ based on backward differencing in time. Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$ and $t^n = n\Delta t$, $n \in \mathbb{Z}$. Also let $\varphi^n = \varphi(x,t^n)$, and $d_t\,\varphi^n = (\varphi^{n+1}-\varphi^n)/\Delta t$. Denote the

approximation of s by W and u by Z. Assuming W^n and Z^n to be known, we determine W^{n+1} and Z^{n+1} as follows :

(3.3.5) $(d_t Z_x^n, X_x)+((a(Z^n)Z_x^{n+1})_x, X_{xx})=a(0)Z_x^n(1)(xZ_x^{n+1}, X_{xx})$

with Z^o defined approximately

and

(3.3.6) $d_t W^n = -a(0)Z_x^{n+1}(1)\ W^{n+1}$

with $W^o = 1.$

Moreover, the discrete Galerkin form of (3.2.12) is given by

(3.3.7) $d_t\ \tau_h^n = (W^{n+1})^2$

with $\tau_h^o = 0.$

3.4 Associated Projection and Related Estimates.

Set

(3.4.1) $A(u;v,w)=((a(u)v_x)_x, w_{xx})-a(0)u_x(1)(xv_x, w_{xx})$, for $u \in W^{1,\infty}(I)$, v and $w \in H^2(I)$.

For the operator A the following Lemma holds.

Lemma 3.4.1 (i) There is a constant M > 0, depending on K_1 and $||u_x||_{L^\infty}$ such that

(3.4.2) $|A(u;v,w)| \leq M||v||_2||w||_2$, $u \in W^{1,\infty}$, v and $w \in H^2$.

(ii) Gårding type inequality holds that is; there exist constants $\tilde{\alpha}$ and ρ such that

(3.4.3) $A(u;v,v) \geq \tilde{\alpha}\ ||v||_2^2 - \rho ||v||_1^2$, for $u \in W^{1,\infty}$ and $v \in H^2(I) \cap H_o^1(I)$, where ρ may depend on K_1 and $||u_x||_{L^\infty}$.

Proof: From (3.4.1), we get

$$|A(u;v,w)| \leq |(a(u)v_{xx}, w_{xx})| + |(a_u(u)u_x v_x, w_{xx})| + |a(0)u_x(1)(xv_x, w_{xx})|$$

$$\leq K_1(||v_{xx}||\ ||w_{xx}|| + ||u_x||_{L^\infty}||v_x||\ ||w_{xx}|| + |u_x(1)|\ ||v_x||\ ||w_{xx}||$$

$$\leq M(K_1, ||u_x||_{L^\infty})\ ||v||_2 ||w||_2.$$

For (ii), we have

$$A(u;v,v) \geq (a(u)v_{xx}, v_{xx}) - \{|(a_u(u)u_x v_x, v_{xx})| + |a(0)u_x(1)(xv_x, v_{xx})|\}$$

$$\geq \alpha ||v_{xx}||^2 - 2K_1 ||u_x||_{L^\infty} ||v_x||\ ||v_{xx}||.$$

Now applying the inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{1}{2}\varepsilon b^2$ for the second term and noting $||\varphi|| \leq ||\varphi_x||$, for $\varphi \in H_o^1$ we get the required result.

Let $\tilde{u} \in \overset{o}{S}_h$ be the projection of u with respect to the form A that is

(3.4.4) $$A(u;u-\tilde{u}, \chi) = 0, \quad \chi \in \overset{o}{S}_h.$$

We shall consider the existence and uniqueness of $\tilde{u}$ subsequently.

Let $L(u)$ and $L^*(u)$ be the one dimensional elliptic operator and formal adjoint associated with (3.4.4), where

$$(3.4.5) \qquad L(u)w = \frac{\partial}{\partial x}(a(u)w_x) - a(u(1))u_x(1)xw_x$$

and

$$(3.4.6) \qquad L^*(u)w = \frac{\partial}{\partial x}(a(u)w_x) + a(u(1))\, u_x(1)(xw)_x.$$

Standard regularity theory combined with uniqueness results, see Douglas et al. [18] modified for one dimensional case imply that the Dirichlet problem for L has atmost one weak solution in $H^1(I)$. Following Douglas [16], one shows that for $\Psi \varepsilon L^2(I)$ there exists a unique solution of the Dirichlet problem

$$L^*(u)\xi = \Psi, \quad x \varepsilon I$$

$$\xi = 0, \quad x = 0 \text{ and } x = 1,$$

satisfying the following regularity condition

$$||\xi||_2 \leq C_o||\Psi||,$$

where the constant C_o depends on u and its derivatives.

For $\Psi \varepsilon L^2(I)$, define $\varphi \varepsilon H^4(I) \cap H_o^1(I)$ by

$$(3.4.7) \qquad L^*(u)\ \varphi_{xx} = \Psi, \quad x \varepsilon I$$

$$\varphi_{xx}\big|_{x=0} = \varphi_{xx}\big|_{x=1} = 0.$$

By regularity assumption and boundary condition (3.4.7) (ii)

$$||\varphi_{xx}||_2 \leq C_o||\Psi||.$$

Since $\varphi \varepsilon H_o^1(I)$, the regularity of the Dirichlet problem for the Laplace operator implies that

(3.4.8) $$||\varphi||_4 \le C_o||\psi||.$$

Integration by parts shows that

(3.4.9) $$(v_1, L^*(u)\varphi_{xx}) = (L(u)v_{1,xx}, \varphi_{xx})$$

$$= A(u; v_1, \varphi), \text{for } v_1 \in H^2(I) \cap H_o^1(I).$$

Now, with the above preparation, we show the existence of $\tilde{u}$ defined in (3.4.4), see Douglas [15].

<u>Lemma</u> 3.4.2. There exists a unique solution $\tilde{u} \in \overset{o}{S}_h$ of (3.4.4), atleast for sufficiently small h.

<u>Proof</u>. It is sufficient to show uniqueness. Let $\tilde{u}_1$ and $\tilde{u}_2$ be two solution of (3.4.4) and set $\theta = \tilde{u}_1 - \tilde{u}_2$. Let $\varphi \in H^4 \cap H_o^1$ satisfy

$$L^*(u)\ \varphi_{xx} = -\theta_{xx},\ x \in I,$$

$$\varphi_{xx}\big|_{x=0} = \varphi_{xx}\big|_{x=1} = 0.$$

Multiplying both the sides by θ and integrating by parts, we use (3.4.9) to get

$$||\theta_x||^2 = (\theta, L^*(u)\varphi_{xx}) = (L(u)\theta_{xx}, \varphi_{xx})$$

$$= A(u; \theta, \varphi).$$

Since $A(u;\theta, X) = 0$, $X \in \overset{o}{S}_h$ and $||\theta|| \le ||\theta_x||$, for $\theta \in H_o^1$ we obtain

$$||\theta||_1^2 \le A(u;\theta, \varphi - X)$$

$$\le M||\theta||_2 \inf_{X \in \overset{o}{S}_h} ||\varphi - X||_2$$

$$\leq K_oMh^2||\theta||_2||\varphi||_4.$$

Applying regularity property that is $||\varphi||_4 \leq C_o||\theta||_2$ we get

$$||\theta||_1^2 \leq K_oC_oMh^2||\theta||_2^2.$$

Since $\theta \in \overset{o}{S}_h$, $A(u;\theta,\theta) = 0$ and by Gårding type inequality (3.4.3)

$$\tilde{\alpha}||\theta||_2^2 \leq \rho||\theta||_1^2$$

$$\leq K_oC_oM\rho h^2||\theta||_2^2.$$

Thus,

$\theta = 0$, for sufficiently small h.

Hence the result.

Let $\eta = u-\tilde{u}$. At first we obtain some estimates of η and its temporal derivative η_t for our later use.

The following Lemma will prove very convenient for that purpose. An analogous result was used by Arnold and Douglas [4] in the context of L^2-Galerkin procedure.

<u>Lemma</u> 3.4.3. Let $\Phi \in H^2(I) \cap H_o^1(I)$ and satisfy

(3.4.10) $\qquad A(u;\Phi,\chi) = F(\chi),\ \chi \in \overset{o}{S}_h,$

where $F : H^2(I) \cap H_o^1(I) \to R$ and linear. Let there exist constants M_1 and M_2 with $M_1 \geq M_2$, such that

$$|F(\varphi)| \leq M_1||\varphi||_2,\ \varphi \in H^2(I) \cap H_o^1(I)$$

(3.4.11) and

$$|F(\varphi)| \leq M_2||\varphi||_4,\ \varphi \in \overset{o}{H}{}^4(I),$$

where $\overset{o}{H}{}^4(I) = \{u \in H^4(I) \cap H_o^1(I) : u_{xx}|_{x=0} = u_{xx}|_{x=1} = 0\}$. Then, for sufficiently small h

$$||\Phi||_2 \leq K_3 [M_1 + \inf_{\chi \in \overset{o}{S}_h} ||\Phi - \chi||_2]$$

(3.4.12) and

$$||\Phi|| \leq K_3 [(M_1 + \inf_{\chi \in \overset{o}{S}_h} ||\Phi - \chi||_2) h^2 + M_2],$$

where $K_3 = K_3(\alpha, \rho, C_o, K_o, M)$ is used as a generic constant.

Proof. First note that

$$A(u;\Phi,\Phi) = A(u;\Phi,\Phi-\chi) - F(\Phi-\chi) + F(\Phi)$$

$$\leq M||\Phi||_2 \inf_{\chi \in \overset{o}{S}_h} ||\Phi-\chi||_2 + M_1 \inf_{\chi \in \overset{o}{S}_h} ||\Phi-\chi||_2 + M_1 ||\Phi||_2.$$

By G$\mathring{a}$rding type inequality (3.4.3), we have

$$\tilde{\alpha}||\Phi||_2^2 - \rho||\Phi||_1^2 \leq (M||\Phi||_2 + M_1) \inf_{\chi \in \overset{o}{S}_h} ||\Phi-\chi||_2 + M_1 ||\Phi||_2.$$

Using the interpolation inequality for $||\Phi||_1$ that is

$||\Phi||_1^2 \leq ||\Phi|| \; ||\Phi||_2$, we get

(3.4.13) $$||\Phi||_2 \leq (\tilde{\alpha})^{-1} [\rho||\Phi|| + M \inf_{\chi \in \overset{o}{S}_h} ||\Phi-\chi||_2 + (K_o+1) M_1].$$

For L^2-estimates, let us recall Aubin-Nitsche's duality argument. For a given $\Psi \in L^2(I)$, define $\varphi \in \overset{o}{H}{}^4(I)$ by (3.4.7). Multiplying both the sides by Φ, we obtain

$$(\Phi,\Psi) = (\Phi, L^*(u)\varphi_{xx}) = A(u;\Phi,\varphi)$$

$$= \inf_{\chi \in \overset{\circ}{S}_h} [A(u;\Phi,\varphi-\chi) - F(\varphi-\chi) + F(\varphi)]$$

$$\leq (M||\Phi||_2+M_1) \inf_{\chi\varepsilon \overset{\circ}{S}_h} ||\varphi-\chi||_2 + M_2||\varphi||_4$$

$$\leq [(M||\Phi||_2+M_1)K_o h^2 + M_2] \, ||\varphi||_4.$$

Applying regularity property (3.4.8), we have

$$(\Phi,\Psi) \leq C_o \, [K_o(M||\Phi||_2+M_1)h^2+M_2] \, ||\Psi||.$$

Thus,

$$(3.4.14) \quad ||\Phi|| \leq C_o \, [K_o(M||\Phi||_2 + M_1)h^2 + M_2].$$

Substituting (3.4.14) in (3.4.13), we get

$$||\Phi||_2 \leq (\tilde{\alpha})^{-1} \, [\rho\{C_oK_o(M||\Phi||_2+M_1)h^2+C_oM_2\}+M \inf_{\chi\varepsilon \overset{\circ}{S}_h} ||\Phi-\chi||_2 + (K_o+1)M_1].$$

For sufficiently small h, we get the required estimate

(3.4.12) (i) for $||\Phi||_2$.

Now, making use of $||\Phi||_2$ in (3.4.14), we obtain the estimate for $||\Phi||$.

<u>Remark</u>. Since $\Phi \varepsilon H^1_o \cap H^2$, the estimate $||\Phi||_1$ can be found out either by interpolation inequality between $||\Phi||$ and $||\Phi||_2$ or directly substituting Φ_{xx} for Ψ and using regularity $||\varphi||_4 \leq C_o||\Phi||_2$.

The next Theorem contains the error analysis of η and η_t for the associated projection (3.4.4).

Theorem 3.4.4. For $t \in [0,T]$, the error $\eta = u-\tilde{u}$ satisfies

(3.4.15) $||\eta||_j \leq K_4\ h^{m-j}||u||_m$, $j = 0,1,2$, $2 \leq m \leq r+1$

and

(3.4.16) $||\eta_t||_j \leq K_5 h^{m-j}(||u_t||_m + ||u||_m)$, $j = 0,1,2$, $2 \leq m \leq r+1$,

where the constants $K_4 = K_4(K_o,K_3)$ and $K_5 = K_5(K_o,K_1,K_3,K_4,$
$||u_x||_{L^\infty(L^\infty)}, ||u_{tx}||_{L^\infty(L^\infty)})$.

Proof. To get the estimate (3.4.15), for $j = 0$ and 2, put $\Phi = \eta$ and $F \equiv 0$ in the previous Lemma 3.4.3, then

$$\begin{aligned} ||\eta||_2 &\leq K_3 \inf_{\chi \in \overset{o}{S}_h} ||\eta - \chi||_2 \\ &\leq K_3 \inf_{v \in \overset{o}{S}_h} ||u-v||_2 \quad , \quad \text{for } v = \chi + \tilde{u} \in \overset{o}{S}_h \\ &\leq K_4\ h^{m-2}\ ||u||_m \quad , \quad \text{for } 2 \leq m \leq r+1. \end{aligned}$$

Similarly,

$$||\eta|| \leq K_4\ h^m ||u||_m, \qquad 2 \leq m \leq r+1.$$

For $j = 1$, the result (3.4.15) follows by interpolation.
To estimate η_t, we differentiate (3.4.4) with respect to 't' and obtain,

(3.4.17) $A(u;\eta_t,\chi) = -(\frac{\partial}{\partial x}([\frac{d}{dt}\ a(u)]\ \eta_x),\ \chi_{xx})$

$$+\ a(0)\ u_{xt}(1)\ (x\ \eta_x,\ \chi_{xx}).$$

Identifying the right hand side of (3.4.17) with $F(\chi)$, we note that for $\varphi \in H^2 \cap H_o^1$

(3.4.18) $\quad |F(\varphi)| \leq K_6 ||\eta||_2 \; ||\varphi||_2,$

where K_6 depends on K_1, $||u_{tx}||_{L^\infty(L^\infty)}$, $||u_x||_{L^\infty(L^\infty)}$.

Further, when $\varphi \in \overset{o}{H}{}^4(I)$, integrating by parts the right hand side of (3.4.17) for $\chi = \varphi$ we obtain

$$F(\varphi) = (a(u)_t \; \eta_x, \; \varphi_{xxx}) - a(0)u_{xt}(1)(\eta, (x\,\varphi_{xx})_x)$$

$$= -(\eta, (a(u)_t \; \varphi_{xxx})_x) - a(0)\; u_{xt}(1)(\eta, (x\,\varphi_{xx})_x).$$

So,

(3.4.19) $\quad |F(\varphi)| \leq K_6 ||\eta|| \; ||\varphi||_4.$

Thus, Lemma 3.4.3 is applicable and we get

$$||\eta_t||_2 \leq K_3 \, [K_6 ||\eta||_2 + \inf_{\chi \in \overset{o}{S}_h} ||\eta_t - \chi||_2]$$
$$\leq K_5 h^{m-2}(||u||_m + ||u_t||_m).$$

Here, we have used the estimate (3.4.15) for j = 2 and the approximation property

$$\inf_{\chi \in \overset{o}{S}_h} ||\frac{\partial \eta}{\partial t} - \chi||_2 = \inf_{\in \overset{o}{S}_h} ||\frac{\partial u}{\partial t} - v||_2 \leq K_o \; h^{m-2} ||u_t||_m, \quad 2 \leq m \leq r+1.$$

Further, using Lemma 3.4.3 and the estimate (3.4.15) for j = 0 we get

$$||\eta_t|| \leq K_3 \, [(K_6 ||\eta||_2 + \inf_{\chi \in \overset{o}{S}_h} ||\eta_t - \chi||_2) h^2 + K_6 ||\eta||].$$

Hence, the required estimate for $||\eta_t||$ is obtained.

For $j = 1$, the estimate $||\eta_t||_1$ follows as usual by interpolation.

We shall also need later the following estimate for $\eta_x(1,t)$.

Theorem 3.4.5. There exists a constant $K_7 = K_7(\alpha, M, K_0, K_4)$ such that

(3.4.20) $$|\eta_x(1,t)| \le K_7 h^{2(m-2)} ||u||_m, \quad 2 \le m \le r+1.$$

Proof. Let us define an auxiliary function $\varphi \in H^4(I) \cap H_0^1(I)$ as a solution of the following problem

$$L^*(u) \; \varphi_{xx} = 0, \quad x \in I;$$

$$\varphi_{xx}|_{x=0} = 0,$$

$$\varphi_{xx}|_{x=1} = 1.$$

Multiplying both the sides of the first equation by η and integrating by parts, we get

$$\begin{aligned} |a(0)\, \eta_x(1,t)| &= |A(u;\eta,\varphi)| \\ &= |A(u;\eta,\varphi-\chi)| \\ &\le M||\eta||_2 \inf_{\chi \in \overset{o}{S}_h} ||\varphi-\chi||_2 \\ &\le K_0 M h^{m-2} ||\eta||_2 \, ||\varphi||_m. \end{aligned}$$

Now applying the estimate (3.4.15) for $j = 2$ and noting $a(\cdot) \ge \alpha$, we have

$$|\eta_x(1,t)| \leq \alpha^{-1} K_o M K_4 h^{2(m-2)} ||\varphi||_m ||u||_m, \quad 2 \leq m \leq r+1,$$

which is the required result.

3.5 A Priori Error Estimates for Continuous Time Galerkin Approximations.

Throughout this section we assume that there are constants K^* and h_o such that a Galerkin approximation $u^h \in \overset{o}{S}_h$ defined by (3.3.2) exists and satisfies

(3.5.1) $$||u^h||_{L^\infty(H^2(I))} \leq K^*, \text{ for } 0 < h \leq h_o,$$

where $u^h(x,0) = Q_h g$

and $A(g;g-Q_h g, X) = 0, \; X \in \overset{o}{S}_h.$

Clearly, $u^h(x,0) \equiv \tilde{u}(x,0).$

We use below the standard notations.

$$e = u-u^h, \; \zeta = u^h-\tilde{u} \text{ and } \eta = u-\tilde{u}, \text{ then } e = \eta - \zeta.$$

Theorem 3.5.1. Suppose $\eta = u-\tilde{u}$ satisfies (3.4.4) and u^h, defined by (3.3.2) is the Galerkin approximation of u. Assume further that (3.5.1) holds, then there is a constant $K_8 = K_8(\alpha, \rho, K^*, K_1, K_4, K_5, K_7)$ such that

(3.5.2) $$|||\zeta|||_{L^\infty(H^1)} + \tilde{\alpha}|||\zeta|||_{L^2(H^2)} \leq K_8 h^{m-1}(|||u_t|||_{L^2(H^m)}$$

$$+ |||u|||_{L^2(H^m)}), \quad 3 \leq m \leq r+1.$$

Proof. From (3.4.4) and (3.3.1) with $v = X$, we get

$$(\tilde{u}_{tx}, \chi_x) + A(u;\tilde{u},\chi) = -(\eta_{tx}, \chi_x), \quad \chi \in \overset{\circ}{S}_h.$$

Subtract it from (3.3.2) to obtain

$$(3.5.3) \quad (\zeta_{tx}, \chi_x) + A(u^h;u^h,\chi) - A(u;\tilde{u},\chi) = (\eta_{tx}, \chi_x).$$

But

$$A(u^h;u^h,\chi)-A(u;\tilde{u},\chi) = A(u;\zeta,\chi) - (\tfrac{\partial}{\partial x}(a(u)-a(u^h))u_x^h, \; \chi_{xx})$$

$$- ((a(u)-a(u^h))u_{xx}^h, \; \chi_{xx}) + a(0)e_x(1)(xu_x^h, \; \chi_{xx}).$$

Substituting this in (3.5.3) and setting $\chi = \zeta$, we have

$$(3.5.4) \quad \frac{1}{2}\frac{d}{dt}\,||\zeta_x||^2 + A(u;\zeta,\zeta) = (\eta_{tx}, \zeta_x) + (\tfrac{\partial}{\partial x}(a(u)-a(u^h))u_x^h, \; \zeta_{xx})$$

$$+ ((a(u)-a(u^h))u_{xx}^h, \zeta_{xx}) - a(0)\;\eta_x(1)(xu_x^h, \zeta_{xx})$$

$$+ a(0)\;\zeta_x(1)(xu_x^h, \zeta_{xx}).$$

Applying Gårding type inequality (3.4.3) for $A(u;\zeta,\zeta)$ and (3.5.1), we integrate by parts the first term in the right hand side of (3.5.4) to get

$$\frac{1}{2}\frac{d}{dt}||\zeta_x||^2 + \tilde{\alpha}||\zeta||_2^2 - \rho||\zeta||_1^2 \leq ||\eta_t||\;\;||\zeta_{xx}|| + K_9(K_1, K^*, K_2)(||\eta||_1 +$$

$$||\zeta||_1)||\zeta_{xx}|| + K_1K^*|\eta_x(1)|\;\;||\zeta_{xx}|| + K_1K^*|\zeta_x(1)|\;\;||\zeta_{xx}||.$$

Using (1.3.15) for $\zeta_x(1)$ and $||\chi|| \leq ||\chi_x||$, for $\chi \in \overset{\circ}{S}_h$, we have

$$(3.5.5) \quad \frac{d}{dt}||\zeta||_1^2 + 2\tilde{\alpha}||\zeta||_2^2 \leq 2\rho||\zeta||_1^2 + 2||\eta_t||\;\;||\zeta||_2 + 2K_9(||\eta||_1 + \eta_x(1))$$

$$||\zeta||_2 + 4K_9||\zeta||_1||\zeta|| + K_1K^*||\zeta||_1^{1/2}||\zeta||_2^{3/2}.$$

From which, we obtain by using $ab \le \frac{(\varepsilon a)^p}{p} + \frac{1}{q}(\frac{b}{\varepsilon})^q (\frac{1}{p} + \frac{1}{q} = 1,$ $a, b, \varepsilon > 0)$

$$\frac{d}{dt} ||\zeta||_1^2 + 2\tilde{\alpha}||\zeta||_2^2 \le K_{10}(||\eta_t||^2 + ||\eta||_1^2 + |\eta_x(1)|^2) + \tilde{K}_{10}(\varepsilon)||\zeta||_2^2$$
$$+ (K_{10} + 2\rho)\ ||\zeta||_1^2.$$

Now with appropriate choice of ε, $\tilde{K}_{10}(\varepsilon)$ can be made equal to $\tilde{\alpha}$. With this choice of ε, we get by integrating with respect to t' from 0 to t and using Gronwall's Lemma (Lemma 1.3.9) the following

$$||\zeta||_1^2(t) + \tilde{\alpha}\int_0^t ||\zeta||_2^2 dt' \le K_{10}\ \exp\ [(K_{10} + 2\rho)t] \int_0^t (||\eta_t||^2 + ||\eta||_1^2$$
$$+ |\eta_x(1)|^2)\ dt' ,$$

where K_{10} depends on K^*, K_1 and K_2.

Now taking supremum over all $t \in [0, T]$ and using the estimates (3.4.15), (3.4.16) and (3.4.20), we obtain

$$||\zeta||_{L^\infty(H^1)} + \beta||\zeta||_{L^2(H^2)} \le K'_{10}\{K_5 h^m(||u_t||_{L^2(H^m)} + ||u||_{L^2(H^m)})$$
$$+ K_4 h^{m-1}||u||_{L^2(H^m)} + K_7 h^{2(m-2)}||u||_{L^2(H^m)}\}.$$

Therefore, for $2(m-2) \ge m-1$ that is for $m \ge 3$, we get the desired estimate (3.5.2).

<u>Corollary</u> 3.5.2. Let the inverse hypothesis (1.3.9) be satisfied for $X \in \overset{o}{S}_h$ alongwith the assumptions in the previous Theorem. Then,

$$(3.5.6) \qquad ||\zeta||_{L^\infty(H^2)} \le K_{11} h^{m-2} \left[||u_t||_{L_2(H^m)} + ||u||_{L^2(H^m)} \right],$$

where the constant $K_{11} = K_{11}(K_o, K_8)$.

Proof. For $\zeta \in \overset{o}{S}_h$, we get by inverse hypothesis (1.3.9)

$$||\zeta_{xx}||_{L^\infty(L^2)} \le K_o h^{-1} ||\zeta_x||_{L^\infty(L^2)}.$$

Thus, (3.5.6) follows immediately using the estimates (1.3.14) and (3.5.2).

From Theorems 3.4.4, 3.5.1 and Corollary 3.5.2, we get the following Theorem.

Theorem 3.5.3. Let the solution u of (3.2.8) - (3.2.10) be sufficiently smooth to satisfy the regularity conditions R_1. Further, suppose that there are positive constants h_o and $K^*(K^* \ge 2K)$ such that an approximate solution $u^h \in \overset{o}{S}_h$ satisfying (3.5.1) exists in $I \times (0,T]$, for $0 < h \le h_o$. Then the following estimates hold for $r \ge 2$

$$(3.5.7) \qquad ||e||_{L^\infty(H^1)} \le K_{12}\, h^r$$

and

$$(3.5.8) \qquad ||e||_{L^\infty(H^2)} \le K_{13}\, h^{r-1},$$

where $K_{12} = K_{12}(K_2, K_4, K_8)$ and $K_{13} = K_{13}(K_2, K_4, K_{11})$ are constants. Further, for sufficiently small h and $r \ge 2$

$$(3.5.9) \qquad ||u^h||_{L^\infty(H^2)} \le 2K_2 \le K^*$$

and consequently, K_{12} as well as K_{13} can be chosen independent of K^*.

Proof. The estimates (3.5.7) and (3.5.8) are immediate from Theorems 3.4.4, 3.5.1 and Corollary 3.5.2 by triangle inequality.

To see (3.5.9), we note

$$\|u^h\|_{L^\infty(H^2)} \le \|u\|_{L^\infty(H^2)} + \|e\|_{L^\infty(H^2)}$$

$$\le K_2 + K_{13}\, h^{r-1}$$

$$\le 2K_2, \text{ for sufficiently small } h \text{ and } r \ge 2.$$

Finally, the Galerkin approximation of the solution $U(y,\tau)$ of the original problem can be defined as

(3.5.10) $$U^h(y,\tau) = u^h(x,t)$$

where

(3.5.11) $$y = s_h(t)\, x$$

(3.5.12) $$\tau = \tau_h(t)$$

where s_h and τ_h are defined by (3.3.3) and (3.3.4) respectively.

Theorem 3.5.4. Under the assumptions of Theorem 3.5.3, (3.2.5) condition B and the regularity hypothesis $\tilde{R}_1$, the following estimates hold for $r \ge 2$

(3.5.13) $$\|S-S_h\|_{L^\infty((0,T_0))} = O(h^r);$$

(3.5.14) $$\|\tau - \tau_h\|_{L^\infty((0,T_0))} = O(h^r)$$

and

$$(3.5.15)\quad |||U-U^h|||_{L^\infty(0,T_o;H^j(\tilde{\Omega}(\tau)))} = O(h^{r+1-j}),\ j = 1,2,$$

where the last norm is understood as in (2.7.23) – (2.7.24).

Proof. Subtracting (3.3.3) from (3.2.11) and then integrating with respect to t, we get

$$|s-s_h| \le K_1 \left[\int_0^t (|\eta_x(1)|+|\zeta_x(1)|)|s|dt' + \int_0^t |u_x^h(1)|\ |s-s_h|\ dt'\right].$$

By applications of Gronwall's Lemma (Lemma 1.3.9) and the inequality (1.3.16) for $\zeta_x(1)$, we get using the estimates (3.4.20) for m = r+1 and (3.5.2),

$$\begin{aligned} ||s-s_h||_{L^\infty(0,T)} &\le K_{14}\{||\eta_x(1,.)||_{L^2(0,T)} + ||\zeta_x(1,.)||_{L^2(0,T)}\} \\ &\le K_{14}\{K_7 h^{2(r-1)}\ ||u||_{L^2(H^{r+1})} + ||\zeta||_{L^2(H^2)}\} \\ &\le K_{15}\ h^r,\ \text{for } r \ge 2. \end{aligned}$$

The estimate (3.5.13) follows immediately, if we note that

$$||S-S_h||_{L^\infty(0,T_o)} = ||s-s_h||_{L^\infty(0,T)}.$$

Moreover, the estimate (3.5.14) follows easily using (3.5.13).

Finally, we have

$$|||U-U^h|||_{L^\infty(0,T_o;H^j(\tilde{\Omega}(\tau)))} \le ||u-u^h||_{L^\infty(0,T;H^j(I))}.$$

Hence, using Theorem 3.5.3, the required estimate for j = 1,2 follows.

3.6 Global Existence and Uniqueness of the Galerkin Approximation.

Now we consider the problem of existence of the Galerkin approximation u^h in the domain of existence of u. Towards this end, let us recall (3.5.3), (3.5.4) and get

$$(3.6.1)\quad (\zeta_{tx},\chi_x)+A(u;\zeta,\chi)=(\eta_{tx},\chi_x)+(\frac{\partial}{\partial x}[a(u)-a(u^h)]u_x^h,\chi_{xx})$$

$$+([a(u)-a(u^h)]u_{xx}^h,\chi_{xx})-a(0)\eta_x(1)(xu_x^h,\chi_{xx})$$

$$+a(0)\zeta_x(1)(xu_x^h,\chi_{xx}).$$

But

$$a(u)-a(u^h) = -\tilde{a}_u\, e = -\int_0^1 \frac{\partial a}{\partial u}(u-\xi e)\, e\, d\xi\,.$$

Replacing u^h by $u-e$ in (3.6.1) and with minor adjustments, we obtain

$$(3.6.2)\quad (\zeta_{xt},\chi_x)+A(u;\zeta,\chi)=(\eta_{tx},\chi_x)-(\tilde{a}_u(\eta-\zeta)(u_{xx}-e_{xx}),\chi_{xx})$$

$$-(\tilde{a}_u(\eta_x-\zeta_x)(u_x-e_x),\chi_{xx})-(\int_0^1 \frac{\partial^2 a}{\partial u^2}(u-\xi e)(u_x-\xi e_x)(\eta-\zeta)$$

$$(u_x-e_x)d\xi\,,\chi_{xx})-a(0)\,\eta_x(1)(x(u_x-e_x),\chi_{xx})$$

$$+a(0)\zeta_x(1)(x(u_x-e_x),\chi_{xx}).$$

Substituting e by $E(x,t)$ for some function $E\in H^2\cap H_0^1$, it follows that

$$(3.6.3)\quad (\zeta_{xt},\chi_x)+A(u;\zeta,\chi)=(\eta_{tx},\chi_x)-(\int_0^1 \frac{\partial a}{\partial u}(u-\xi E)(\eta-\zeta)(u_{xx}-E_{xx})d\xi,$$

$$\chi_{xx})-(\int_0^1 \frac{\partial a}{\partial u}(u-\xi E)(\eta_x-\zeta_x)(u_x-E_x)d\xi, \chi_{xx})-(\int_0^1 \frac{\partial^2 a}{\partial u^2}(u-\xi E)$$

$$(u_x-\xi E_x)(\eta-\zeta)(u_x-E_x)d\xi, \chi_{xx})-a(0)\eta_x(1)(x(u_x-E_x),\chi_{xx})$$

$$+ a(0)\zeta_x(1)(x(u_x-E_x),\chi_{xx}), \ \chi \in \overset{o}{S}_h.$$

This is a linear ordinary differential equation in ζ. Therefore, for any $E = E(x,t)$ there exists a unique solution ζ of (3.6.3) with

(3.6.4) $$\zeta(x,0) = 0$$

in the interval $(0,T]$.

The equation (3.6.3) defines an operator $\mathcal{J}$ such that $\zeta = \mathcal{J}E$ for each $E \in H^2 \cap H_0^1$. Since $e = \eta-\zeta$, therefore

(3.6.5) $$e = \eta - \mathcal{J}E, \text{ for each } E \in H^2 \cap H_0^1.$$

To show the existence of a solution u^h for (3.3.2), we need to show that the operator equation (3.6.5) has a fixed point. In otherwords, we are looking for an $e(E)$ such that $e(E) = E$.

Theorem 3.6.1. Assume that the finite element space $\overset{o}{S}_h$ satisfies the inverse property (1.3.9) and u is the unique solution of (3.2.8) - (3.2.10). More over, let the condition B (3.2.5) and the regularity hypothesis R_1 for u be satisfied. Then for $\delta > 0$, there exists a solution $u^h \in \overset{o}{S}_h$ of (3.3.2) satisfying $||u-u^h||_{L^\infty(0,T;H^2)} \leq \delta$.

Proof. Choose $X = \zeta$ in (3.6.3) to obtain

$$\frac{1}{2}\frac{d}{dt}||\zeta_x||^2+\tilde{\alpha}||\zeta||_2^2-\rho||\zeta||_1^2 \le ||\eta_t||\;||\zeta_{xx}||+K_1(||\eta||_{L^\infty}+||\zeta||_{L^\infty})$$

$$(||u_{xx}||+||E_{xx}||)||\zeta_{xx}||+K_1(||\eta_x||+||\zeta_x||)(||u_x||_{L^\infty}+||E_x||_{L^\infty})$$

$$||\zeta_{xx}||+K_1(||u_x||_{L^\infty}+||E_x||_{L^\infty})^2(||\eta||+||\zeta||)||\zeta_{xx}||+K_1|\eta_x(1)|$$

$$(||u_x||+||E_x||)||\zeta_{xx}||+K_1(||u_x||+||E_x||)|\zeta_x(1)|\;||\zeta_{xx}||.$$

Using (1.3.16) for $\zeta_x(1)$ and applying Young's inequality for this term and some other terms, we get

$$\frac{d}{dt}||\zeta||_1^2+2\tilde{\alpha}||\zeta||_2^2 \le K_{16}(\varepsilon)||\zeta||_2^2+K_{17}(\varepsilon,K_1,K_2)\{||\eta_t||^2+$$

$$(||\eta||^2_{L^\infty}+||\eta||_1^2)(1+||E||_2^2)+||\eta||^2(1+2||E||_2^2(1+||E||_2^2))+$$

$$|\eta_x(1)|^2(1+||E||_2^2)\}+K_{18}(\varepsilon,K_1,K_2)\{||E||_2^2(1+||E||_2^2)+\rho\}||\zeta||_1^2.$$

Choosing ε appropriately so that $K_{16}(\varepsilon) = 2\tilde{\alpha}$, integrating with respect to t and then applying Gronwall's Lemma (Lemma 1.3.9), we obtain

$$||\zeta||_1^2(t) \le K_{17}\exp\left[K_{18}\{||E||^2_{L^\infty(H^2)}(1+||E||^2_{L^\infty(H^2)})+\rho\}t\right]\int_0^t[||\eta_t||^2$$

$$+||\eta||_1^2(1+||E||^2_{L^\infty(H^2)})+|\eta_x(1)|^2(1+||E||^2_{L^\infty(H^2)})$$

$$+||\eta||^2\{1+2||E||^2_{L^\infty(H^2)}(1+||E||^2_{L^\infty(H^2)})\}]\,dt'.$$

From Theorem 3.4.4 and Theorem 3.4.5, we get

$$(3.6.6) \quad \|\zeta\|_{L^\infty(H^1)} \le K_{19}\,[h^{r+1}+(1+\|E\|_{L^\infty(H^2)})(h^r+h^{2(r-1)})$$

$$+\, h^{r+1}\{1+\|E\|_{L^\infty(H^2)}\,(1+\|E\|_{L^\infty(H^2)})\}]\ ,$$

where K_{19} depends on K_2, K_{17}, K_{18}, $\|E\|_{L^\infty(H^2)}$ and ρ.

Thus, we have,

$$(3.6.7) \quad \|e\|_{L^\infty(H^2)} \le \|\eta\|_{L^\infty(H^2)} + \|\zeta\|_{L^\infty(H^2)}$$

$$\le \|\eta\|_{L^\infty(H^2)} + K_o h^{-1}\|\zeta\|_{L^\infty(H^1)}.$$

Now, for $\|E\|_{L^\infty(H^2)} \le \delta$ we get from (3.6.6) and (3.6.7)

$$\|e\|_{L^\infty(H^2)} \le K_{20}\, h^{r-1}\ ,$$

where $K_{20} = K_{20}(K_{19}, K_o, K_4, \rho, \delta)$.

Therefore, for sufficiently small h and $r \ge 2$

$$\|e\|_{L^\infty(H^2)} \le \delta\ .$$

It is easily verified, see Theorem 2.7.2 that the operator is continuous on finite dimensional space and hence compact. So the operator $E \to e$ has a fixed point by Schauder's fixed point theorem (Theorem 1.3.11). Hence the result.

The uniqueness can be proved following the arguments in Theorem 2.5.2, except that the estimate of $\varphi_x(1,t)$ is given by the (1.3.16) instead of (1.3.15).

We formalize the above in the form of a Theorem.

Theorem 3.6.2. Let u satisfy the regularity condition R_1 and let $K > 0$. Then there exists one and only one solution $u^h \in \overset{o}{S}_h$ of (3.3.2) in the ball $\{||u-u^h||_{L^\infty(H^2)} \leq K\}$, for sufficiently small h and $r \geq 2$.

3.7 A Priori Error Estimates for the Discrete Time Galerkin Approximations.

In this Section, we first develop a priori bounds for the error u^n-Z^n and s^n-W^n. The results obtained will be similar to those obtained for the continuous case in Section 3.5. We assume that condition B (3.2.5) and the regularity hypothesis R_2 for $\{u,s\}$ hold. Let $\eta^n = u^n-\tilde{u}^n$, $\zeta^n = Z^n-u^n$ and $e^n = \eta^n-\zeta^n = u^n-Z^n$.

Theorem 3.7.1. Let Z^0 be determined by

$$(3.7.1) \qquad A(g;g-Z^0,X) = 0.$$

There exist constants $K_{21} = K_{21}(\rho,\alpha,M,K_0,K_1,K_2,K_4,K_5$ and $K_7)$ and $h_0 > 0$ such that if $h \leq h_0$, $t = o(h)$ and $r \geq 2$

$$(3.7.2) \qquad \sup_{t^n} ||\zeta||_1^2+\beta\Delta t \sum_{n=0}^{N-1} ||\zeta^{n+1}||_2^2 \leq K_{21}\{(\Delta t)^2+h^{2r}\}.$$

Proof. For $t = t^{n+1}$, the projection (3.4.4) is given by

$$(3.7.3) \qquad A(u^{n+1};\eta^{n+1},X) = 0, \; X \in \overset{o}{S}_h.$$

From (3.7.3) and (3.3.1) with $v = X$, for $t = t^{n+1}$ we get

$$(3.7.4)\quad (d_t\tilde{u}_x,\chi_x)+A(u^{n+1},\tilde{u}^{n+1},\chi) = -(d_t\ \eta^n_x,\chi_x)+(d_t u^n_x-(\frac{\partial u_x}{\partial t})^{n+1},\chi_x).$$

Subtracting (3.7.4) from (3.3.5), we obtain

$$(3.7.5)\quad (d_t\,\zeta^n_x,\chi_x)+A(Z^n,Z^{n+1},\chi)-A(u^{n+1},\tilde{u}^{n+1},\chi)$$

$$= (d_t\ \eta^n_x,\chi_x) - (d_t u^n_x-(\frac{\partial u_x}{\partial t})^{n+1},\chi_x).$$

But,

$$(3.7.6)\quad A(Z^n,Z^{n+1},\chi)-A(u^{n+1},\tilde{u}^{n+1},\chi)=A(u^{n+1},\zeta^{n+1},\chi)-(\frac{\partial}{\partial x}([a(u^n)-$$

$$a(Z^n)]\ Z^{n+1}_x),\chi_{xx})-a(0)e^n_x(1)(xZ^{n+1}_x,\chi_{xx})-(\frac{\partial}{\partial x}(\ a(u^{n+1})$$

$$-a(u^n)]Z^{n+1}_x),\chi_{xx})-a(0)\,\Delta t\ d_t\ u^n_x(1)(xZ^{n+1}_x,\chi_{xx})\ .$$

Integrating by parts the last two terms on the right hand side of (3.7.5) and using (3.7.6) in (3.7.5), we get by choosing $\chi = \zeta^{n+1}$

$$(3.7.7)\quad (d_t\,\zeta^n_x,\zeta^{n+1}_x)+A(u^{n+1},\zeta^{n+1},\zeta^{n+1})= -(d_t\eta,\zeta^{n+1}_{xx})+(d_t u^n-(\frac{\partial u}{\partial t})^{n+1},$$

$$\zeta^{n+1}_{xx})+([a(u^n)-a(Z^n)]\ Z^{n+1}_{xx},\zeta^{n+1}_{xx})+(a_u(Z^n)e^n_xZ^{n+1}_x,\zeta^{n+1}_{xx})$$

$$+([a_u(u^n)-a_u(Z^n)]\ u^n_xZ^{n+1}_x,\zeta^{n+1}_{xx})+a(0)\ \eta^n_x(1)(xZ^{n+1}_x,\zeta^{n+1}_{xx})$$

$$-a(0)\,\zeta^n_x(1)(xZ^{n+1}_x,\zeta^{n+1}_{xx})+([a(u^{n+1})-a(u^n)]\ Z^{n+1}_{xx},\zeta^{n+1}_{xx})$$

$$+(a_u(u^{n+1})\ \Delta t\ d_t u^n_xZ^{n+1}_x,\zeta^{n+1}_{xx})+([a_u(u^n)-a_u(u^{n+1})]\,u^n_xZ^{n+1}_x,$$

$$\zeta^{n+1}_{xx})+a(0)\ \Delta t\ d_t\ u^n_x(1)(xZ^{n+1}_x,\zeta^{n+1}_{xx}).$$

Since, $(d_t \zeta_x^n, \zeta_x^{n+1}) \geq \frac{1}{2\Delta t}(||\zeta_x^{n+1}||^2 - ||\zeta_x^n||^2)$ and

$A(u^{n+1}, \zeta^{n+1}, \zeta^{n+1}) \geq \tilde{\alpha}||\zeta^{n+1}||_2^2 - \rho||\zeta^{n+1}||_1^2$ we have from (3.7.7)

$$(3.7.8)\quad \frac{1}{2\Delta t}(||\zeta_x^{n+1}||^2 - ||\zeta_x^n||^2) + \tilde{\alpha}||\zeta^{n+1}||_2^2 - \rho||\zeta^{n+1}||_1$$
$$\leq ||d_t \eta^n||\ ||\zeta_{xx}^{n+1}|| + ||\sigma_n||\ ||\zeta_{xx}^{n+1}|| + K_1||Z_{xx}^{n+1}||$$
$$(||\zeta^n||_{L^\infty} + ||\eta^n||_{L^\infty})||\zeta_{xx}^{n+1}|| + K_1(||\zeta_x^n|| + ||\eta_x^n||)||Z_x^{n+1}||_{L^\infty}$$
$$||\zeta_{xx}^{n+1}|| + K_1(||\zeta^n|| + ||\eta^n||)||u_x^n||_{L^\infty}||Z_x^{n+1}||_{L^\infty}||\zeta_{xx}^{n+1}||$$
$$+ K_1|\eta_x^n(1)|\ ||Z_x^{n+1}||\ ||\zeta_{xx}^{n+1}|| + K_1|\zeta_x^n(1)|\ ||Z_x^{n+1}||$$
$$||\zeta_{xx}^{n+1}|| + K_1\ \Delta t||d_t u^n||_{L^\infty}||Z_{xx}^{n+1}||\ ||\zeta_{xx}^{n+1}|| + K_1\ \Delta t||d_t u_x^n||$$
$$||Z_x^{n+1}||_{L^\infty}||\zeta_{xx}^{n+1}|| + K_1||d_t u^n||\ ||u_x^n||_{L^\infty}||Z_x^{n+1}||_{L^\infty}\Delta t$$
$$||\zeta_{xx}^{n+1}|| + K_1\ \Delta t|d_t u_x^n(1)|\ ||Z_x^{n+1}||\ ||\zeta_{xx}^{n+1}||,$$

where $\quad \sigma_n = d_t u^n - (\frac{\partial u}{\partial t})^{n+1}.$

Using generalized Young's inequality (Inequality III of Chapter 1) for $K_1|\zeta_x^n(1)|\ ||Z_x^{n+1}||\ ||\zeta_{xx}^{n+1}||$, using $|\zeta_x^n(1)| \leq ||\zeta_x^n|| + ||\zeta_x^n||^{1/2}$ $||\zeta_{xx}^n||^{1/2}$ and the inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2}b^2$ for remaining terms we get

$$(3.7.9)\quad \frac{1}{2\Delta t}(||\zeta^{n+1}||_1^2 - ||\zeta^n||_1^2) + \tilde{\alpha}||\zeta^{n+1}||_2^2 \leq \rho||\zeta^{n+1}||_1^2 + \frac{\varepsilon}{4}||\zeta^n||_2^2$$
$$+ (7 + \tfrac{1}{4})\ \varepsilon\ ||\zeta^{n+1}||_2^2 + K(K_1, K_2, \varepsilon)||Z^{n+1}||_2^2(1 + ||Z^{n+1}||_2^2)\ .$$

$$||\zeta^n||_1^2+K(K_1,K_2,\varepsilon)||Z^{n+1}||_2^2(||\eta^n||_1^2+|\eta_x^n(1)|^2)+$$

$$K(\varepsilon)||d_t\,\eta^n||^2+K(K_1,K_2,\varepsilon)\ \Delta t^2||Z^{n+1}||_2^2(||d_tu^n||_{L^\infty}^2+$$

$$||d_tu_x^n||_{L^2}^2 + |d_tu_x^n(1)|^2) + K(\varepsilon)\ ||\sigma_n||^2.$$

We next multiply each of the terms of (3.7.9) by Δt and sum on n, $n = 0,1,\ldots,J-1$ to obtain

$$(3.7.10)\quad \frac{1}{2}\ ||\zeta^J||_1^2+\Delta t\ \tilde{\alpha}\sum_{n=0}^{J-1}||\zeta^{n+1}||_2^2 \le \Delta t\rho\sum_{n=0}^{J-1}||\zeta^n||_1^2+\Delta t\,\rho||\zeta^J||_1^2$$

$$+\ \Delta tK(K_1,K_2,\varepsilon)\sum_{n=0}^{J-1}\ ||Z^{n+1}||_2^2(1+||Z^{n+1}||_2^2)||\zeta^n||_1^2$$

$$+\ (7+\frac{1}{4})\varepsilon\ \Delta t\sum_{n=0}^{J-1}||\zeta^{n+1}||_2^2+\frac{\varepsilon}{4}\ \Delta t\sum_{n=0}^{J-1}||\zeta^n||_2^2$$

$$+\ K(K_1,K_2,\varepsilon)\ \Delta t\sum_{n=0}^{J-1}||Z^{n+1}||_2^2(||\eta^n||_1^2+|\eta_x^n(1)|^2)$$

$$+K(\varepsilon)\ \Delta t\sum_{n=0}^{J-1}||d_t\eta^n||^2+\Delta t^3K(K_1,K_2,\varepsilon)\sum_{n=0}^{J-1}||Z^{n+1}||_2^2$$

$$(||d_tu^n||_{L^\infty}^2+||d_tu_x^n||_{L^2}^2+|d_tu_x^n(1)|^2)+K(\varepsilon)\Delta t\sum_{n=0}^{J-1}||\sigma_n||^2.$$

But,

$$(3.7.11)\quad \Delta t\sum_{n=0}^{N-1}||\sigma_n||^2 \le (\Delta t)^2\ \left|\left|\frac{\partial^2 u}{\partial t^2}\right|\right|_{L^2(H^1)}^2,$$

$$(3.7.12)\quad \Delta t\sum_{n=0}^{N-1}||d_tu^n||_1^2 \le C||u_t||_{L^2(H^1)}^2$$

and

$$(3.7.13)\quad \Delta t\sum_{n=0}^{N-1}|d_tu_x^n(1)|^2 \le C\ ||u_{xt}||_{L^2(L^\infty)}^2.$$

Choose ε and Δt in such a way that $\tilde{\alpha} - 15\varepsilon > \alpha_1$ and $\frac{1}{2} - \Delta t\,\rho = \alpha_2$ in (3.7.10), then we get using (3.7.11) - (3.7.13)

$$(3.7.14)\quad \alpha_2 \|\zeta^J\|_1^2 + 2\Delta t\ \alpha_1 \sum_{n=0}^{J} \|\zeta^n\|_2^2 \le \Delta t \sum_{n=0}^{J-1} \{\rho + K(K_1,K_2)\|Z^{n+1}\|_2^2 (1+\|Z^{n+1}\|_2^2)\} \|\zeta^n\|_1^2 + K(K_1,K_2)\Delta t \sum_{n=0}^{J-1} \|Z^{n+1}\|_2^2 (\|\eta^n\|_1^2 + |\eta_x^n(1)|^2 + \|d_t\ \eta^n\|^2) + (\Delta t)^2 K(K_1,K_2) \sup_{0\le n\le J} \|Z^n\|_2^2 (\|u_t\|^2_{L^2(H^1)} + \|u_t\|^2_{L^2(W^{1,\infty})}) + (\Delta t)^2 \|u_{tt}\|^2_{L^2(H^1)}.$$

Now assume that there is a $K^* \ge 2K_2$ such that

$$(3.7.15)\qquad \|Z^n\|_2 \le K^*,\ \text{for } n = 0,1,\ldots,J.$$

Then, applying (3.7.15), (3.4.15), $j = 1$, (3.4.16), $j = 0$ and (3.4.20) we get

$$\alpha_1 \|\zeta^J\|_1^2 + \Delta t\ \alpha_1 \sum_{n=0}^{J} \|\zeta^n\|_2^2 \le \Delta t \sum_{n=0}^{J-1} \{\rho + K(K_1,K_2)K^{*2}(1+K^{*2})\} \|\zeta\|_1^2 + K(K_1,K_2,K^*)\{h^{2r} + h^{4(r-1)}\} + K(K_1,K_2,K^*)(\Delta t)^2 + K_2(\Delta t)^2.$$

Finally, applying discrete Gronwall's Lemma (Lemma 1.3.10) we get for $r \ge 2$

$$(3.7.16)\quad \|\zeta^J\|_1^2 + \Delta t\ \beta \sum_{n=0}^{J} \|\zeta^n\|_2^2 \le K_{21}(K^*)\ \{h^{2r} + (\Delta t)^2\}.$$

Now we show that for sufficiently small h, K_{21} is independent of K^*. Applying inverse property (1.3.9) we get for $t = t^n$, $n = 0,1,2,\ldots,J$,

$$||z^n||_2 \leq ||e^n||_2 + ||u^n||_2$$

$$\leq ||\eta^n||_2 + ||\zeta^n||_2 + ||u^n||_2$$

$$\leq Kh^{r-1} + K_o h^{-1}||\zeta^n||_1 + K_2$$

$$\leq Kh^{r-1} + K_o h^{-1} K_{22}(K^*) (h^r+\Delta t) + K_2.$$

So, for sufficiently small h and $\Delta t = o(h)$,

$$(3.7.17) \qquad \sup_{0 \leq n \leq J} ||z^n||_2 \leq 2K_2.$$

Therefore, the required result follows.

<u>Corollary</u> 3.7.2. Suppose that all the assumptions of Theorem 3.7.1 hold and that the space and time discretizations satisfy the relation $\Delta t = o(h)$. Further, suppose that $\overset{o}{S}_h$ satisfies the inverse property (1.3.9). Then there is a constant $K_{21}'(K_o, K_{21})$ such that for $r \geq 2$

$$\sup_n ||\zeta_{xx}^n||^2 \leq K_{21}'\{h^{-2}(\Delta t)^2 + h^{2r-2}\}.$$

This follows from the inverse property (1.3.9) and the estimate (3.7.2).

The Theorem to follow shows the estimates of e and e_1, where $e_1 = s-W$ at discrete time level $t = t^n$.

<u>Theorem</u> 3.7.3. There exists K_{23} depending on K_{21}, K_4, K_7, K_1 and K_2 such that for $r \geq 2$

$$(3.7.18) \quad \sup_{t^n} (||e||_1^2 + |e_1|^2) \leq K_{23}\{(\Delta t)^2 + h^{2r}\}.$$

<u>Proof</u>. The estimate for $\|e^n\|_1$, for $n = 0,1,\ldots,N$ follows from Theorems 3.7.1, 3.4.4 and triangle-inequality.

To find an estimate for e_1, consider (3.2.11) for $t = t^{n+1}$ and obtain

$$(3.7.19) \qquad d_t s^n = d_t s^n - (\tfrac{ds}{dt})^{n+1} - a(0)u_x^{n+1}(1)\ s^{n+1}.$$

Subtracting (3.3.6) from (3.7.19) and multiplying the resulting one by e_1^{n+1}, we get

$$(3.7.20) \quad \langle d_t e_1^n, e_1^{n+1}\rangle = \langle d_t s^n - (\tfrac{ds}{dt})^{n+1}, e_1^{n+1}\rangle - \langle a(0) Z_x^{n+1}(1) e_1^{n+1}, e_1^{n+1}\rangle$$
$$- \langle a(0) e_x^{n+1}(1) s^{n+1}, e_1^{n+1}\rangle$$

with $\quad e_1^0 = 0.$

Here $\langle u,v\rangle = uv$. Now the left hand side of (3.7.20) is estimated by

$$(3.7.21) \qquad \langle d_t e_1^n, e_1^{n+1}\rangle \geq \frac{1}{2\Delta t}\,(|e_1^{n+1}|^2 - |e_1^n|^2).$$

From (3.7.20) - (3.7.21), we have

$$\frac{1}{2\Delta t}(|e_1^{n+1}|^2 - |e_1^n|^2) \leq |\sigma_{1,n}|\ |e_1^{n+1}| + K_1 |Z_x^{n+1}(1)|\ |e_1^{n+1}|^2$$
$$+ K_1 K_2 |\eta_x^{n+1}(1)|\ |e_1^{n+1}| + K_1 K_2 |\zeta_x^{n+1}(1)|\ |e_1^{n+1}|,$$

where $\sigma_{1,n} = d_t s^n - (\frac{ds}{dt})^{n+1}$.

Using Sobolev Imbedding Theorem (Proposition 1.3.7) for one dimensional domain for $\zeta_x(1)$ and applying the inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2} b^2$, $a,b \geq 0$; $\varepsilon > 0$, we get summing on n, $n=0,1,\ldots,J-1$,

$$(3\cdot7\cdot22)\quad \frac{1}{2\Delta t}\,|e_1^J|^2 \leq \sum_{n=0}^{J-1}|\sigma_{1,n}|^2+3\sum_{n=0}^{J-1}|e_1^{n+1}|^2+K_1\sum_{n=0}^{J-1}|Z_x^{n+1}(1)|\,|e_1^{n+1}|^2$$

$$+K(K_1,K_2)\sum_{n=0}^{J-1}|\eta_x^{n+1}(1)|^2+K(K_1,K_2)\sum_{n=0}^{J-1}||\zeta^{n+1}||_2^2\,.$$

Multiplying both the sides by Δt and applying the estimate $(3\cdot7\cdot2)$ for the last term in $(3\cdot7\cdot22)$, as well as $(3\cdot4\cdot20)$ for $\eta_x(1)$, we get by $(3\cdot7\cdot17)$

$$(3\cdot7\cdot22)\ \frac{1}{2}\,|e_1^J|^2 \leq (\Delta t)^2||s_{tt}||^2_{L^2(0,T)}+3\Delta t\sum_{n=0}^{J-1}|e_1^n|^2+2K_1K_2\ \Delta t$$

$$\sum_{n=0}^{J-1}|e_1^n|^2+\Delta t(3+2K_1K_2)|e_1^J|^2+K(K_1,K_2,K_7)h^{4(r-1)}$$

$$+\ K(K_1,K_2,K_{21})\ h^{2r}.$$

For small Δt, $\frac{1}{2}-\Delta t\ (3+2K_1K_2)=\beta_1$ and now applying discrete Gronwall's Lemma (Lemma $1\cdot3\cdot10$), we get the required estimate.

The following is a direct consequence.

<u>Corollary</u> 3.7.4. Suppose that all the assumptions in Theorem 3.7.3 hold and that $\overset{o}{S}_h$ satisfies the inverse property $(1\cdot3\cdot9)$. Then there is a constant $K'_{23}=K(K_0,K_4,K_{23})$ such that for $\Delta t=o(h)$,

$$\sup_{t^n}||e||_2^2 \leq K'_{23}\{h^{-2}(\Delta t)^2+h^{2r-2}\}\,.$$

We are now looking for a discrete time Galerkin approximation of $\{U,S\}$, where $\{U,S\}$ is the solution of $(3\cdot2\cdot1)-(3\cdot2\cdot4)$. The approximation U_h^n is given by

$$U_h^n(y_h,\tau_h) \equiv U_h(\tau_h^n) = Z(x,t^n),$$

where τ_h^n is defined by (3.3.7) and $y_h^n = xW^n$.

Similarly the discrete time Galerkin approximation S_h^n is defined by

$$S_h^n \equiv S_h(\tau_h^n) = W(t^n).$$

Remark. Here τ_h^n corresponds to t^n under the approximate transformation. Consequently, the time mesh in τ_h^n corresponding to a uniform time mesh Δt may not be uniform.

Let $\Delta\tau_h^n = \tau_h^{n+1}-\tau_h^n$ and $\Delta\tau_h = \max\limits_{0\le n\le N} \Delta\tau_h^n$.

Finally, we establish the following theorem for the discrete time error estimates $U^n-U_h^n$, $S^n-S_h^n$, where $U^n \equiv U(\tau_h^n)$, $S^n \equiv S(\tau_h^n)$ and τ^n is the time corresponding to t^n in the original transformation.

Theorem 3.7.5. Suppose $\tilde{R}_2$ holds, $r \ge 2$ and that the free boundary S is bounded away from zero that is there is a positive constant ν such that $S \ge \nu$ for all $\tau \in [0,T]$. Then the following estimate holds for a backward difference in time,

$$(3.7.23) \quad \sup_n \{ \|U^n-U_h^n\|_{H^1(\tilde{\Omega}^n)} + |S^n-S_h^n| \} = O(\Delta\tau_h + h^r),$$

where the norm $\|\cdot\|_{H^j(\tilde{\Omega}^n)}$ is understood in the usual sense and

$$\tilde{\Omega}^n = (0,\ \min(S^n, S_h^n,\ S(\tau^n))).$$

Proof. Since $\sup\limits_{t^n} |s^n-W^n| = O(\Delta t+h^r)$ and $s^n \ge \nu$, there exists

an $\varepsilon > 0$ such that $W^n \geq \nu - \varepsilon = \nu_1 > 0$ (say), for $n = 0,1,\ldots,N$.

From (3.3.7) $\frac{\Delta\tau_h^n}{\Delta t} = (W^{n+1})^2$, we have $\Delta\tau_h^n \geq \nu_1^2 \Delta t$ for $n=0,1,\ldots,N$

and hence $\Delta\tau_h \geq \nu_1^2 \Delta t$ that is $\Delta\tau_h = O(\Delta t)$.

Now $|S^n - S_h^n| \leq |S(\tau^n) - S_h^n| + |S^n - S(\tau^n)| \leq |e_1^n| + K_2|\tau^n - \tau_h^n|$.

Further,

$$||U^n - U_h^n||_{H^1(\tilde{\Omega}^n)} \leq ||U(\tau^n) - U_h^n||_{H^1(\tilde{\Omega}^n)} + ||U^n - U(\tau^n)||_{H^1(\tilde{\Omega}^n)}$$

$$\leq ||u^n - Z^n||_1 + K_2|\tau^n - \tau_h^n|.$$

Thus,

$$(3.7.24) \quad \sup_n \{||U^n - U_h^n||_{H^1(\tilde{\Omega}^n)} + |S^n - S_h^n|\} \leq K(K_2, K_{23})\{\Delta t + h^{r+1-j} + \sup_n |\tau^n - \tau_h^n|\}.$$

Now we estimate $\sup |\tau^n - \tau_h^n|$. From (3.2.12) for $t = t^{n+1}$

$$(3.7.25) \quad d_t\, \tau^n = d_t\, \tau^n - \left.\frac{d\tau}{dt}\right|_{t=t^{n+1}} + (s^{n+1})^2.$$

Subtracting (3.3.7) from (3.7.25), we get

$$(3.7.26) \quad d_t(\tau^n - \tau_h^n) = d_t\, \tau^n - \left.\frac{d\tau}{dt}\right|_{t^{n+1}} + e_1^{n+1}(s^{n+1} + W^{n+1}).$$

Since,

$$|d_t(\tau^n - \tau_h^n)| \geq \frac{1}{\Delta t}(|\tau^{n+1} - \tau_h^{n+1}| - |\tau^n - \tau_h^n|)$$

and $\sum_{n=0}^{N} (d_t\tau^n - \left.\frac{d\tau}{dt}\right|_{t^{n+1}}) \leq K\, \Delta t \sum_{n=0}^{N} ||\tau_{tt}||_{L^2(t^n, t^{n+1})}$

$$\leq K(K_2)||s_t||_{L^2(0,T)},$$

we have, summing on $n = 0, \ldots, J-1$, using the estimate (3.7.18) and multiplying by Δt, the following estimate

$$(3.7.27) \quad |\tau^J - \tau_h^J| \leq K(K_2)(\Delta t + h^r).$$

Taking supremum over all J, $(0 \leq J \leq N)$, substituting this in (3.7.24), we get the desired result as $\Delta\tau_h = O(\Delta t)$.

<u>Corollary</u> 3.7.6. Suppose that all the assumptions in the Theorem 3.7.5 hold and that $\overset{\circ}{S}_h$ satisfies inverse property. Then, the following estimate for $\Delta t = o(h)$

$$(3.7.28) \quad \sup_n \{ ||U^n - U_h^n||_{H^2(\tilde{\Omega}^n)} \} = O(h^{-1}\,\Delta\tau_h + h^{r-1})$$

holds.

<u>Proof</u>. Now,

$$||U^n - U_h^n||_{H^2(\tilde{\Omega}^n)} \leq ||e^n||_2 + ||U^n - U(\tau^n)||_{H^2(\tilde{\Omega}^n)}$$

$$\leq ||e^n||_2 + K(K_2)\; |\tau^n - \tau_h^n|.$$

From the Corollary 3.7.4, $\Delta\tau_h = O(\Delta t)$ and the estimate (3.7.27), we get the desired estimate (3.7.28) because $\Delta\tau_h = o(h)$.

CHAPTER 4

STEFAN PROBLEM WITH A QUASILINEAR PARABOLIC EQUATION IN NON DIVERGENCE FORM

4.1 Introduction.

In this Chapter, we deal with a single phase Stefan problem with a quasilinear parabolic equation in non-divergence form. As usual, we use a coordinate transformation to fix the domain and then apply H^1-Galerkin procedure. Optimal error estimates in L^2, H^1 as well as H^2-norms are obtained for this problem.

In Section 2, the discription of the problem and transformed problem are given. The weak formulation and H^1-Galerkin procedure are discussed in Section 3. Section 4 deals with an auxiliary projection and some approximation Lemmas. Optimal error estimates in L^2, H^1 and H^2-norms for continuous time Galerkin approximations are derived, assuming existence of the approximate solution in Section 5. Finally in Section 6 the question of global existence and uniqueness of the Galerkin approximation is discussed.

4.2 Problem Description and Domain Fixing.

We now consider the following nonlinear Stefan problem in a variable domain $\Omega(\tau) \times (0,T_0]$ as given in Chapter 3. Find a pair $\{U,S\}$, $U = U(y,\tau)$, $S = S(\tau)$ such that U satisfies

(4.2.1) $\quad U_\tau - a(U)U_{yy} = 0, \ (y,\tau) \in \Omega(\tau) \times (0,T_o]$

with initial and boundary conditions

(4.2.2) $\quad U(y,0) = g(y), \ y \in I;$

(4.2.3) $\quad \left.\begin{array}{l} U_y(0,\tau) = 0, \\ U(S(\tau),\tau) = 0, \end{array}\right. \quad \tau > 0,$

and $S(\tau)$, the free boundary satisfies

(4.2.4) $\quad S_\tau = -U_y(S(\tau),\tau), \ \tau > 0$

with $\quad S(0) = 1.$

Here, the following conditions are assumed on $a(\cdot)$, g and on the pair $\{U,S\}$ and are termed collectively as 'condition B'.

CONDITION B.

(i) For $p \in R$, $a(p) \geq \alpha$, where α is a positive constant.

(ii) For $p \in R$, $a(p) \in C^3(R)$ and there is a common bound $K_1 > 0$ such that

$$|a|, \ |a_p|, \ |a_{pp}| \text{ and } |a_{ppp}| \leq K_1.$$

(iii) The initial function g is sufficiently smooth and satisfies the compatibility condition that is

$$g_y(0) = g(1) = 0.$$

(iv) The problem (4.2.1) - (4.2.4) has a unique solution.

Further, assume that the solution $\{U,S\}$ of (4.2.1) - (4.2.4) satisfies the following regularity condition :

$\tilde{R}_1$. $U \varepsilon L^\infty(0,T_0;H^{r+1}(\Omega(\tau))) \cap L^\infty(0,T_0;H^2(\Omega(\tau)))$

$\cap W^{1,2}(0,T_0;H^{r+1}(\Omega(\tau))) \cap W^{1,\infty}(0,T_0;W^{2,\infty}(\Omega(\tau)))$

$L^\infty(0,T_0;W^{2,\infty}(\Omega(\tau)))$, and

$S \in W^{1,\infty}(0,T_0)$.

Let $\tilde{K}_2$ be the bound for the functions in above mentioned norms.

In order to fix the free boundary, we use below a co-ordinate transformation

(4.2.5) $$x = S^{-1}(\tau)y.$$

Further, a time scale transformation

(4.2.6) $$t = t(\tau) = \int_0^\tau S^{-2}(\tau')\, d\tau'$$

is used to decouple the resulting transformed system.

Following 2.3, an routine calculation shows that the function $u(x,t) = U(y,\tau)$ satisfies

(4.2.7) $$u_t - a(u)u_{xx} = -u_x(1)\, xu_x,\ x \in I,\ t \in (0,T]\ ;$$

(4.2.8) $$u(x,0) = g(x),\ x \in I;$$

(4.2.9) $$u_x(0,t) = u(1,t) = 0,\ t > 0$$

and the function $s(t) = S(\tau)$ satisfies

(4.2.10) $$\frac{ds}{dt} = -u_x(1)s,\ t > 0$$

with $$s(0) = 1.$$

Here $t = T$ corresponds to $\tau = T_o$. Note that all the regularity assumptions for $\{U, S\}$ carry over to $\{u, s\}$ with the bound say, K_2 and the new regularity assumptions be collectively called R_1. Further, the integral (4.2.6) can be rewritten as

$$(4.2.11) \quad \frac{d\tau}{dt} = s^2(t) \text{ with } \tau(0) = 0.$$

4.3 <u>The Weak Formulation and H^1-Galerkin Procedure.</u>

Consider the space :

$$\overset{o}{H}{}^2(I) = \{v \in H^2(I) : v_x(0) = v(1) = 0\}.$$

The weak solution of (4.2.7) - (4.2.9) is given by

$$(4.3.1) \quad (u_{tx}, v_x) + (a(u)u_{xx}, v_{xx}) = u_x(1)(xu_x, v_{xx}), v \in \overset{o}{H}{}^2(I).$$

<u>H^1-Galerkin procedure.</u> Let $\overset{o}{S}_h \subset \overset{o}{H}{}^2(I)$ be a finite dimensional subspace belonging to regular $\overset{o}{S}{}_h^{r,2}$ family. Let $\overset{o}{S}_h$ satisfy the approximation property (1.3.6) with $k = 2$ and inverse property (1.3.9). Now we call $u^h : (0,T] \to \overset{o}{S}_h$ an H^1-Galerkin approximation of u if it satisfies

$$(4.3.2) \quad (u^h_{tx}, \chi_x) + (a(u^h)u^h_{xx}, \chi_{xx}) = u^h_x(1)(xu^h_x, \chi_{xx}), \ \chi \in \overset{o}{S}_h$$

and the initial condition

$$(4.3.3) \qquad u^h(x,0) = Q_h\, g(x),$$

where Q_h is an appropriate projection of u onto $\overset{o}{S}_h$ at $t = 0$ to be defined later.

Further, the Galerkin approximations s_h and τ_h of s and τ respectively are given by

$$(4.3.4) \qquad \frac{ds_h}{dt} = -u_x^h(1)s_h\,, \text{ with } s_h(0) = 1$$

and

$$(4.3.5) \qquad \frac{d\tau_h}{dt} = s_h^2\,, \qquad \text{with } \tau_h(0) = 0.$$

4.4 Some Approximation Lemmas.

Set

$$(4.4.1) \quad A(u;v,w) = (a(u)v_{xx},w_{xx}) - u_x(1)(xv_x,w_{xx}), \text{for } u \in W^{1,\infty},$$
$$v \text{ and } w \in H^2.$$

The boundedness and Gårding type inequality for A can be established (see, Lemma 2.6.1) by standard arguments.

Lemma 4.4.1. For $u \in W^{1,\infty}$, v and $w \in \overset{o}{H}{}^2(I)$,

$$(4.4.2) \qquad |A(u;v,w)| \le M||v_{xx}||\ ||w_{xx}||$$

and

$$(4.4.3) \qquad A(u;v,v) \ge \tilde{\alpha}\ ||v_{xx}||^2 - \rho||v_x||^2,$$

where M, $\tilde{\alpha}$ and ρ are constants, but M, ρ may depend on $||u_x||_{L^\infty}$.

Define

$$(4.4.4) \qquad A_\rho(u;v,w) = A(u;v,w) + \rho(v_x,w_x).$$

Note that $A_\rho(u;\cdot,\cdot)$ is coercive in $\overset{o}{H}{}^2$, that is

$$(4.4.5) \qquad A_\rho(u;v,v) \ge \tilde{\alpha}\ ||v_{xx}||^2.$$

Let $\tilde{u} \in \overset{o}{S}_h$ be an approximation of u with respect to the form A_ρ :

$$(4.4.6) \qquad A_\rho(u;u-\tilde{u},\chi) = 0,\ \chi \in \overset{o}{S}_h.$$

Now, an application of Lax-Milgram Theorem (Theorem 1.3.3) shows the existence and uniqueness of $\tilde{u}$, defined in (4.4.6).

Consider

(4.4.7) $L^*(u)\varphi = \frac{\partial^2}{\partial x^2}(a(u)\varphi_{xx}) + u_x(1)(x\varphi_{xx})_x - \rho\varphi_{xx}, \; u \in \overset{\circ}{H}{}^2(I).$

For $\Psi \in L^2(I)$, define $\varphi \in H^4 \cap \overset{\circ}{H}{}^2$ by

(4.4.8) $$L^*(u)\varphi = \Psi, \; x \in I,$$

$$\varphi_{xx}\big|_{x=1} = \varphi_{xxx}\big|_{x=0} = 0.$$

Then for $v \in \overset{\circ}{H}{}^2(I)$ we get

(4.4.9) $$(v, L^*(u)\varphi) = A_\rho(u; v, \varphi).$$

Thus, defining $D(L^*)$ as

$$D(L^*) = \{\varphi \in H^4 \cap \overset{\circ}{H}{}^2 : \varphi_{xx}(1) = \varphi_{xxx}(0) = 0\},$$

we have from the positivity and boundedness of A_ρ that at least a weak solution $\varphi \in D(L^*)$ of (4.4.8), for each $\Psi \in L^2$ exists. Further, by the regularity Theorem (Theorem 1.3.4), we get

(4.4.10) $$||\varphi||_4 \leq C_0||\Psi||,$$

where C_0 depends on u and its derivatives.

Let $\eta = u - \tilde{u}$. We now need to obtain some estimates of η and its temporal derivatives η_t for our future use. The following Lemma proves very convenient for our purpose.

<u>Lemma</u> 4.4.2. Let $\Phi \in \overset{\circ}{H}{}^2(I)$ and satisfy

In order to get an L^2-estimate, we follow here Aubin-Nitsche's duality arguments. For $\Psi \in L^2(I)$, define $\varphi \in D(L^*)$ by (4.4.8). Multiply both the sides of (4.4.8) by Φ to obtain for $u \in \overset{o}{H}{}^2$,

$$(4.4.16) \quad (\Phi,\Psi) = (\Phi, L^*(u)\varphi) = A_\rho(u;\Phi,\varphi)$$

$$= [A_\rho(u;\Phi,\varphi-X) + F(X-\varphi) + F(\varphi)]$$

$$\leq M\|\Phi_{xx}\| \inf_{X\in \overset{o}{S}_h} \|\varphi-X\|_2 + M_1 \inf_{X\in \overset{o}{S}_h} \|\varphi-X\|_2 + M_2\|\varphi\|_4$$

$$\leq [M\|\Phi_{xx}\|K_o h^2 + M_1 K_o h^2 + M_2]\,\|\varphi\|_4.$$

From (4.4.10), (4.4.14) and (4.4.16), we obtain the required estimate (4.4.15).

The next Lemma contains the error analysis of η and η_t.

Lemma 4.4.3. For $t \in [0,T]$, the following estimates hold for η and η_t

$$(4.4.17) \qquad \|\eta\|_j \leq K_4\, h^{m-j}\, \|u\|_m$$

and

$$(4.4.18) \qquad \|\eta_t\|_j \leq K_5 h^{m-j}(\|u_t\|_m + \|u\|_m),\; j=0,1,2 \text{ and } 2\leq m\leq r+1.$$

Here K_4 and K_5 are positive constants depending on parameter expressed through the following expressions, $K_4 = K_4(K_o,K_3)$ and $K_5 = K_5(K_o,K_1,K_3,K_4,\|u_t\|_{W^{2,\infty}}$ and $\|u\|_{W^{2,\infty}})$.

Proof. Put $\Phi = \eta$ and $F \equiv 0$ in the previous Lemma 4.4.2 to get

$$\|\eta_{xx}\| \leq K_3 \inf_{X\in \overset{o}{S}_h} \|\eta-X\|_2$$

$$\leq K_3 \inf_{v \varepsilon \overset{o}{S}_h} ||u-v||_2, \text{ for } v = \chi + \tilde{u} \varepsilon \overset{o}{S}_h$$

$$\leq K_o K_3 h^{m-2} ||u||_m.$$

For $\eta \varepsilon \overset{o}{H}{}^2$, $||\eta|| \leq ||\eta_x||$ and $||\eta_x|| \leq ||\eta_{xx}||$ and hence the result (4.4.17), for $j = 2$.

Similarly, we get the following estimate for $||\eta||$,

$$||\eta|| \leq K_4 h^m ||u||_m.$$

Consequently, the estimate for $||\eta||_1$ follows from the interpolation inequality,

$$||\eta||_1 \leq ||\eta||^{1/2} \; ||\eta||_2^{1/2}.$$

In order to estimate η_t, we differentiate (4.4.6) with respect to t and obtain

(4.4.19) $\quad A_\rho(u,\eta_t,\chi) = -((\frac{d}{dt}a(u))\ \eta_{xx},\chi_{xx}) + u_{tx}(1)(x\eta_x,\chi_{xx}).$

Identifying the right hand side of (4.4.19) with $F(\chi)$, we see that for $\varphi \varepsilon \overset{o}{H}{}^2(I)$

$$|F(\varphi)| \leq K_6 \; ||\eta_{xx}|| \; ||\varphi_{xx}||,$$

where K_6 depends on K_1 and $||u_t||_{W^{1,\infty}}$.

Further, for $\varphi \varepsilon D(L^*)$ we get on integration by parts

(4.4.20) $\quad F(\varphi) = (\eta_x, (a_t(u)\varphi_{xx})_x) - u_{tx}(1)(\eta, (x\varphi_{xx})_x)$

$$= -(\eta, (a_t(u)\varphi_{xx})_{xx}) + (a_t(u)\,\varphi_{xx})_x \,\Big]_{x=0}^{x=1}$$

$$+ u_{tx}(1)(\eta, (x\varphi_{xx})_x).$$

For $u \in \overset{o}{H}{}^2$ and $\varphi_{xxx}\big|_{x=0} = 0$, the second term in the right hand side of (4.4.20) vanishes. So,

$$|F(\varphi)| \leq K_7 \|\eta\| \, \|\varphi\|_4,$$

where $K_7 = K_7(K_1, \|u_{txx}\|_{L^\infty}, \|u_{xx}\|_{L^\infty}$ and $\|u_t\|_{W^{1,\infty}})$.

Thus, Lemma 4.4.2 is applicable with $M_1 = K_6\|\eta_{xx}\|$ and $M_2 = K_7\|\eta\|$ and we get the desired estimate (4.4.18) for $j = 0, 2$. For $j = 1$, as usual we make use of the interpolation inequality to get the estimate (4.4.18).

We shall also need later the following estimate for $\eta_x(1)$.

<u>Lemma</u> 4.4.4. There is a constant $K_8 = K_8(\alpha, K_0, M, K_4)$ such that for $2 \leq m \leq r+1$,

$$(4.4.21) \qquad |\eta_x(1)| \leq K_8 h^{2(m-2)} \|u\|_m.$$

<u>Proof</u>. Define an auxiliary function $\varphi \in H^4 \cap \overset{o}{H}{}^2$ as a solution of

$$L^*(u)\varphi = 0, \quad x \in I;$$

$$\varphi_{xxx}\big|_{x=0} = 0 ;$$

$$\varphi_{xx}\big|_{x=1} = 1.$$

Multiplying by η the first equation and integrating by parts, we obtain

$$\alpha\, |\eta_x(1)| \leq |a(0)\eta_x(1)| \leq |A_\rho(u;\eta,\varphi)|$$

$$\leq A_\rho(u;\eta,\varphi-\chi), \quad \chi \in \overset{o}{S}_h$$

$$\leq M \, \|\eta\|_2 \inf_{\chi \varepsilon \overset{\circ}{S}_h} \|\varphi - \chi\|_2$$

$$\leq MK_4K_o h^{2(m-2)} \, \|u\|_m \|\varphi\|_m .$$

Hence the result follows.

4.5 A Priori Error Estimates for Continuous Time Galerkin Approximation.

Throughout this Section, we assume that there are positive constants K^* and h_o such that a Galerkin approximation $u^h \varepsilon \overset{\circ}{S}_h$ in (4.3.2) exists and satisfies

(4.5.1) $\quad \|u^h\|_{L^\infty(H^2)} \leq K^*$, for $0 < h \leq h_o$,

where $u^h(x,0)$ is defined as $Q_h g$, satisfying

(4.5.2) $\quad A_\rho(g; g-Q_h g, \chi) = 0$, for $\chi \varepsilon \overset{\circ}{S}_h$.

Clearly, $\quad u^h(x,0) \equiv \tilde{u}(x,0)$.

Let $\zeta = u^h - \tilde{u}$ and $e = u - u^h = \eta - \zeta$.

Theorem 4.5.1. Suppose $\eta = u - \tilde{u}$ satisfies (4.4.6) and u^h, the Galerkin approximation of u is defined by (4.3.2) with Q_h given as in (4.5.2). Further, assume that (4.5.1) holds. Then there is a constant $K_9 = K_9(\alpha, \rho, K^*, K_1, K_4, K_5$ and $K_8)$ such that for $m \geq 4$,

(4.5.3) $\quad \|\zeta_x\|_{L^\infty(L^2)} + \beta \|\zeta_{xx}\|_{L^2(L^2)}$

$$\leq K_9 h^m (\|u_t\|_{L^2(H^2)} + \|u\|_{L^2(H^m)}).$$

Proof. From (4.4.6) and (4.3.1) with $v = \chi$, we get

$$(\tilde{u}_{tx},\chi_x)+A_\rho(u;\tilde{u},\chi) = -(\eta_{tx},\chi_x) + \rho(u_x,\chi_x),\ \chi \in \overset{o}{S}_h.$$

Subtracting this from (4.3.2), we obtain

$$(4.5.4)\quad (\zeta_{tx},\chi_x)+A_\rho(u^h;u^h,\chi)-A_\rho(u;\tilde{u},\chi)=(\eta_{tx},\chi_x)-\rho(\eta_x,\chi_x)+\rho(\zeta_x,\chi_x).$$

But,

$$(4.5.5)\quad A_\rho(u^h;u^h,\chi)-A_\rho(u;\tilde{u},\chi)=(a(u^h)\zeta_{xx},\chi_{xx})+([a(u^h)-a(u)]\tilde{u}_{xx},$$

$$\chi_{xx})-u_x(1)(x\zeta_x,\chi_{xx})+\eta_x(1)(xu^h_x,\chi_{xx})-\zeta_x(1)(xu^h_x,\chi_{xx})+\rho(\zeta_x,\chi_x).$$

From (4.5.4)-(4.5.5) with $\chi = \zeta$ we get on integrating by parts with respect to x the two terms on the right hand side of (4.5.4)

$$\frac{1}{2}\frac{d}{dt}||\zeta_x||^2+(a(u^h)\zeta_{xx},\zeta_{xx})=-(\eta_t,\zeta_{xx})+\rho(\eta,\zeta_{xx})+u_x(1)(x\zeta_x,\zeta_{xx})$$

$$+([a(u)-a(u^h)]\,\tilde{u}_{xx},\zeta_{xx})-\eta_x(1)(xu^h_x,\zeta_{xx})+\zeta_x(1)(xu^h_x,\zeta_{xx}).$$

Using $a(\cdot) \geq \alpha$, (4.5.1) and replacing $\tilde{u}$ by $u-\eta$, we obtain

$$(4.5.6)\quad \frac{1}{2}\frac{d}{dt}||\zeta_x||^2+\alpha||\zeta_{xx}||^2\leq||\eta_t||\ ||\zeta_{xx}||+\rho||\eta||\ ||\zeta_{xx}||$$

$$+K_2||\zeta_x||^2+K_2(||\eta||_{L^\infty}+||\zeta||_{L^\infty})\ ||\eta||_2||\zeta_{xx}||$$

$$+K_1K_2(||\eta||+||\zeta||)||\zeta_{xx}||+K^*|\eta_x(1)|\ ||\zeta_{xx}||$$

$$+K^*|\zeta_x(1)|\ ||\zeta_{xx}||.$$

Since $|\zeta_x(1)| \leq ||\zeta_x||^{1/2}||\zeta_{xx}||^{1/2}$ for $\zeta \in \overset{o}{H}{}^2$, applying Young's inequality for the last term and the inequality $ab\leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2}b^2$,

$a, b \geq 0$; $\varepsilon > 0$ for the remaining terms in (4.5.6) to get using $||\varphi||_{L^\infty} \leq ||\varphi_x||$, for $\varphi \in \overset{o}{H}{}^2$

$$\frac{1}{2}\frac{d}{dt}||\zeta_x||^2+\alpha||\zeta_{xx}||^2 \leq K_{10}(\varepsilon)||\zeta_{xx}||^2+K(K_1,K_2,K^*,\rho,\varepsilon)(||\eta_t||^2$$

$$+||\eta||^2+|\eta_x(1)|^2)+K(K_2,\varepsilon)||\eta||_1^2||\eta||_2^2+K(K_2,\varepsilon)||\eta||_2^2$$

$$||\zeta_x||^2+K(K_1,K_2,K^*,\varepsilon)\ ||\zeta_x||^2.$$

Now with appropriate choice of ε, $K_{10}(\varepsilon)$ can be made less than or equal to $\alpha/2$. With this choice of ε, we get by integrating with respect to 't' and using Gronwall's Lemma (Lemma 1.3.9) the following,

$$||\zeta_x||^2(t)+\alpha\int_0^t||\zeta_{xx}||^2\,dt' \leq K(K_1,K_2,K^*,\rho)\int_0^t(||\eta_t||^2+||\eta||^2$$

$$+\ |\eta_x(1)|^2+||\eta||_1^2||\eta||_2^2)\ dt'.$$

From (4.4.17), (4.4.18) and (4.4.21) with $2(m-2) \geq m$ and $2m-3 \geq m$ that is $m \geq 4$, we get the desired estimate (4.5.3).

<u>Corollary</u> 4.5.2. Let all the assumptions of the Theorem 4.5.1 hold and the finite dimensional space $\overset{o}{S}_h$ satisfy the inverse property (1.3.9). Then there exists a constant K_{11} depending on K_9, K_o such that for $r+1 \geq m \geq 4$,

$$(4.5.7)\quad ||\zeta||_{L^\infty(L^2)}+||\zeta||_{L^\infty(H^1)}+h||\zeta||_{L^\infty(H^2)}$$

$$\leq K_{11}h^m(||u||_{L^2(H^m)}+||u_t||_{L^2(H^m)}).$$

Proof. From the estimate (4.5.3) and $||\zeta|| \leq ||\zeta_x||$ for $\zeta \in \overset{o}{S}_h$ we get,

$$||\zeta||_{L^\infty(L^2)} + ||\zeta||_{L^\infty(H^1)} \leq K_{11} h^m (||u||_{L^2(H^m)} + ||u_t||_{L^2(H^m)}).$$

By inverse property (1.3.9), we have

$$||\zeta_{xx}||_{L^\infty(L^2)} \leq K_o\, h^{-1} ||\zeta_x||_{L^\infty(L^2)}.$$

Hence the result (4.5.7).

From Theorem 4.5.1, Corollary 4.5.2, Lemma 4.4.3 and triangle inequality we get the following Theorem.

Theorem 4.5.3. Let the solution u of (4.2.7) - (4.2.9) satisfy the regularity hypothesis R_1. Further, suppose that there are positive constants h_o and K^*, ($K^* \geq 2K_2$) such that an approximate solution $u^h \in \overset{o}{S}_h$ in (4.3.2) satisfying (4.5.1) exists in $I \times (0,T]$ for $0 < h \leq h_o$. Then the following estimates hold for $r \geq 3$,

(4.5.8) $$||e||_{L^\infty(H^j)} \leq K_{12}\, h^{r+1-j}, \quad j = 0,1,2,$$

where $K_{12} = K_{12}(K_2, K_4$ and $K_{11})$.

Besides, for sufficiently small h and $r \geq 3$

(4.5.9) $$||u^h||_{L^\infty(H^2)} \leq 2K_2 \leq K^*$$

and consequently, K_{12} can be chosen independent of K^*.

Proof. The estimates (4.5.8) for j = 0,1,2 are immediate from the Theorem 4.5.1, Corollary 4.5.2 and Lemma 4.4.3 by triangle inequality.

To prove (4.5.9), we note

$$||u^h||_{L^\infty(H^2)} \leq ||u||_{L^\infty(H^2)} + ||e||_{L^\infty(H^2)}$$

$$\leq K_2 + K_{12}\, h^{r-1}$$

$$\leq 2K_2, \text{ for sufficiently small } h \text{ and } r \geq 3.$$

We are now looking for approximations of U and S where the pair $\{U, S\}$ is the solution of (4.2.1) - (4.2.4). The Galerkin approximations U^h and S_h are given by

(4.5.10) $\qquad U^h(y, \tau) = u^h(x,t);$

(4.5.11) $\qquad S_h(\tau) \quad = s_h(t),$

where

$$y = s_h(t)x,$$

(4.5.12)

$$\tau = \tau_h.$$

and s_h, τ_h are given by (4.3.4), (4.3.5) respectively.

<u>Theorem</u> 4.5.4. Suppose that the condition B and the regularity hypothesis $\tilde{R}_1$ are satisfied. Then the following estimates hold for $r \geq 3$,

(4.5.13) $\qquad ||S-S_h||_{L^\infty(0,T_0)} = O(h^{r+1}),$

(4.5.14) $\qquad ||\tau-\tau_h||_{L^\infty(0,T_0)} = O(h^{r+1})$

and

$$(4.5.15)\quad |||U-U^h|||_{L^\infty(0,T_0;H^j(\tilde{\Omega}(\tau)))} = O(h^{r+1-j}),\; j = 0,1,2,$$

where $|||\cdot|||$ is defined as in (2.7.23) - (2.7.24).

Proof. From (4.2.10) and (4.3.4), we have

$$|s-s_h| \le \int_0^t (|\eta_x(1)|+|\zeta_x(1)|)\ |s|\ dt' + \int_0^t |u_x^h(1)|\ |s-s_h|\ dt'.$$

An application of Gronwall's Lemma (Lemma 1.3.9) and the estimates (4.4.21), (4.5.3) for m = r+1 give

$$(4.5.16)\quad ||s-s_h||_{L^\infty(0,T)} \le K(K_2)\{||\eta_x(1)||_{L^2(0,T)} + ||\zeta_x(1)||_{L^2(0,T)}\}$$

$$\le K(K_2)\ \{K_8 h^{2(r-1)}||u||_{L^2(H^{r+1})} + ||\zeta_{xx}||_{L^2(L^2)}\}$$

$$\le K_{13}\ h^{r+1},\ \text{for } r \ge 3.$$

The estimate (4.5.13) is immediate from (4.5.16), if we note that

$$||S-S_h||_{L^\infty(0,T_0)} = ||s-s_h||_{L^\infty(0,T)}.$$

Further, the estimate (4.5.14) follows from (4.2.11), (4.3.5) and (4.5.16).

Finally, since

$$|||U-U^h|||_{L^\infty(0,T_0;H^j(\tilde{\Omega}(\tau)))} = ||u-u^h||_{L^\infty(0,T;H^j(I))},$$

we obtain the required estimate (4.5.15).

Remark. For optimal rate of convergence in L^2-norm, we can use finite elements consisting C^1-splines of degree $r \ge 3$.

4.6 Global Existence and Uniqueness of the Galerkin Approximation.

Now we consider the problem of existence of the Galerkin approximation u^h in the domain of existence of u. Towards this end, let us recall (4.5.4) and note

$$A_\rho(u^h;u^h,X)-A_\rho(u;\tilde{u},X) = A_\rho(u;\zeta,X)+([a(u^h)-a(u)]\ u^h_{xx},X_{xx})$$
$$+\ \eta_x(1)(xu^h_x,X_{xx})-\zeta_x(1)\ (xu^h_x,X_{xx}).$$

From the above we get,

$$(4.6.1)\quad (\zeta_{xt},X_x) + A_\rho(u;\zeta,X) = (\eta_{tx},X_x)-\rho(\eta_x,X_x)+\rho(\zeta_x,X_x)$$
$$+\ ([a(u)-a(u^h)]\ u^h_{xx},X_{xx})-\eta_x(1)(xu^h_x,X_{xx})$$
$$+\ \zeta_x(1)(xu^h_x,X_{xx}).$$

But,

$$(4.6.2)\quad a(u)-a(u^h) = \tilde{a}_u e = -\int_0^1 \frac{\partial a(u-\xi e)}{\partial u}\ e\ d\xi\ .$$

Replacing u^h by u-e in (4.6.1) with (4.6.2), we have

$$(4.6.3)\quad (\zeta_{tx},X_x)+A_\rho(u;\zeta,X) = -(\eta_t,X_{xx})+\rho(\eta,X_{xx})+\rho(\zeta_x,X_x)$$
$$-\ (\int_0^1 \frac{\partial a(u-\xi e)}{\partial u}(\eta-\zeta)\ d\xi\ (u_{xx}-e_{xx}),\ X_{xx})$$
$$-\ \eta_x(1)(x(u_x-e_x),X_{xx})+\zeta_x(1)(x(u_x-e_x),X_{xx}).$$

Substitute e by E(x,t), where $E \in \overset{o}{H}{}^2$. Then we get,

$$(4.6.4)\quad (\zeta_{tx},X_x)+A_\rho(u;\zeta,X) = -(\eta_t,X_{xx})+\rho(\eta,X_{xx})+\rho(\zeta_x,X_x)$$

$$-\left(\int_0^1 \frac{\partial a}{\partial u}(u-\xi E)(\eta-\zeta)d\xi\ (u_{xx}-E_{xx}),\chi_{xx}\right)-\eta_x(1)(x(u_x-E_x),\chi_{xx})$$

$$+\ \zeta_x(1)(x(u_x-E_x),\chi_{xx})\ \}$$

which is a linear ordinary differential equation in ζ. Therefore, for any $E = E(x,t)$ there exists a unique solution ζ of (4.6.4) with

(4.6.5) $\qquad \zeta(x,0) = 0$

in the interval $(0,T)$.

The equation (4.6.4) defines an operator $\mathcal{J}$ such that $\zeta = \mathcal{J}(E)$, for each $E \varepsilon \overset{o}{H}{}^2$. Since $e = \eta - \zeta$, therefore

(4.6.6) $\qquad e = \eta - \mathcal{J}(E)$, for $E \varepsilon \overset{o}{H}{}^2$.

To show the existence of a solution u^h in (4.3.2), we need to show that the operator equation (4.6.6) has a fixed point. In otherwords, we are looking for an $e(E)$ such that

$$e(E) = E.$$

<u>Theorem</u> 4.6.1. Suppose that the finite element space $\overset{o}{S}_h$ satisfies inverse property (1.3.9) and u is the unique solution of (4.2.7) - (4.2.9). Further, let the regularity condition R_1 be satisfied. Then for some $\delta > 0$, there exists a solution $u^h \varepsilon \overset{o}{S}_h$ of (4.3.2) satisfying $\|u-u^h\|_{L^\infty(0,T;H^2(I))} \leq \delta$.

<u>Proof</u>. Set $\chi = \zeta$ in (4.6.4) to get

$$\frac{1}{2}\frac{d}{dt}\|\zeta_x\|^2+\tilde{\alpha}\|\zeta_{xx}\|^2 \leq \|\eta_t\|\ \|\zeta_{xx}\|+\rho\|\eta\|\ \|\zeta_{xx}\|+\rho\|\zeta_x\|^2$$

$$+ K_1(\|\eta\|_{L^\infty}+\|\zeta\|_{L^\infty})(\|u_{xx}\|+\|E_{xx}\|)\|\zeta_{xx}\|$$

$$+ |\eta_x(1)|\ (\|u_x\|+\|E_x\|)\|\zeta_{xx}\|+(\|u_x\|+\|E_x\|)$$

$$|\zeta_x(1)|\ \|\zeta_{xx}\|.$$

Using $|\zeta_x(1)| \leq \|\zeta_x\|^{1/2}\ \|\zeta_{xx}\|^{1/2}$, applying Young's inequality for the last term and the inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2} b^2, a, b \geq 0, \varepsilon > 0$ for the remaining terms we get,

$$\frac{d}{dt}\|\zeta_x\|^2+2\tilde{\alpha}\|\zeta_{xx}\|^2 \leq K_{14}(\varepsilon)\|\zeta_{xx}\|^2+K(K_1,\rho,K_2,\varepsilon)(\|\eta_t\|^2+\|\eta\|^2_{L^\infty}$$

$$(1+\|E\|_2^2)+|\eta_x(1)|^2(1+\|E\|_2^2))+K(\rho,K_1,K_2,\varepsilon)(1+\|E\|_2^2)$$

$$\|\zeta_x\|^2.$$

Choosing ε appropriately so that $2\tilde{\alpha} = K_{14}(\varepsilon)$, integrating with respect to t and there after applying Gronwall's Lemma (Lemma 1.3.9) we get

$$\|\zeta\|_1^2(t) \leq K(K_1,K_2,\rho)\exp\left[K(\rho,K_1,K_2)t(1+\|E\|^2_{L^\infty(H^2)})\right]\int_0^t\{\|\eta_t\|^2$$

$$+ (|\eta_x(1)|^2 + \|\eta\|_1^2)(1+\|E\|^2_{L^\infty(H^2)})\}.$$

From Lemmas 4.4.3 - 4.4.4, it follows that

$$(4.6.7) \qquad \|\zeta\|_{L^\infty(H^1)} \leq K_{15}\{h^{r+1}+(h^{2(r-1)}+h^r)(1+\|E\|_{L^\infty(H^2)})\},$$

where $K_{15} = K_{15}(K_1,K_2,K_4,K_5,K_8,\rho$ and $\|E\|_{L^\infty(H^2)})$.

Thus, we have

$$(4.6.8) \qquad ||e||_{L^\infty(H^2)} \le ||\eta||_{L^\infty(H^2)} + ||\zeta||_{L^\infty(H^2)}$$

$$\le ||\eta||_{L^\infty(H^2)} + K_o\, h^{-1} ||\zeta||_{L^\infty(H^1)} .$$

For $||E||_{L^\infty(H^2)} \le \delta$ and from (4.6.7) - (4.6.8), we get

$$||e||_{L^\infty(H^2)} \le K_{16}\, h^{r-1},$$

where K_{16} depends on K_{15}, K_4, K_o and δ .

Therefore, for sufficiently small h

$$||e||_{L^\infty(H^2)} \le \delta .$$

Now, an application of Schauder's fixed point theorem (Theorem 1.3.11) guarantees the existence of an E such that e = E, which is a solution of the operator equation (4.6.6).

The uniqueness can be proved following the arguments of Chapter 2. We formalize the above in the form of a Theorem.

Theorem 4.6.2. Let u satisfy the regularity hypothesis R_1 and let $K > 0$. Then there exists one and only one solution $u^h \in \overset{o}{S}_h$ of (4.3.2) in the ball $\{||u-u^h||_{L^\infty(H^2)} \le K\}$, for sufficiently small h and $r \ge 3$.

CHAPTER 5

SINGLE PHASE SEMILINEAR STEFAN PROBLEM

5.1 Introduction.

In this chapter we consider Galerkin methods for approximating the solution of a single phase semilinear Stefan problem for continuous time, backward difference in time as well as extrapolated Crank Nicolson schemes. Optimal rates of convergence in $L^2, H^1, W^{1,\infty}$ and H^2-norms are derived for these cases.

In Section 2, the main problem, the basic assumptions and the question of straightening the free boundary are discussed. The weak formulation, Galerkin procedures and some approximation Lemmas are described in Section 3. Section 4 deals with optimal error estimates in $L^2, H^1, W^{1,\infty}$ and H^2-norms for the continuous time Galerkin approximation. A priori error estimates for the discrete time Galerkin procedures are derived in Section 5.

5.2 Problem Formulation and Basic Assumptions.

A single phase semilinear Stefan problem is stated as follows : Find a pair $\{(U,S) : U = U(y,\tau),\ S = S(\tau)\}$ of functions such that U satisfies the semilinear parabolic equation,

$$(5.2.1) \qquad U_\tau - U_{yy} = f(U), \ (y,\tau) \in \Omega \times (0,T_0] ,$$

subject to the initial and boundary conditions,

$$(5.2.2) \qquad U(y,0) = g(y), \quad y \in I,$$

$$(5.2.3) \qquad \begin{array}{l} U_y(0,\tau) = 0 \;, \\ \qquad\qquad\qquad\qquad \tau > 0, \\ U(S(\tau),\tau) = 0, \end{array}$$

where the free boundary $S(\tau)$ is governed by

$$(5.2.4) \qquad S_\tau + U_y(S(\tau),\tau) = 0, \; \tau > 0$$

with initial condition $S(0) = 1.$

We assume that $f(\cdot)$ is a bounded smooth function on R, $|f|, |f_u| \leq K_1$ and that a pair $\{U,S\}$ of sufficiently smooth solutions exists and is unique.

We organize our regularity hypotheses on the solution $\{U,S\}$ of (5.2.1) - (5.2.4) according to the results in which they are used. We assume

$\tilde{R}_1.$ $U \in L^\infty(0,T_0;H^{r+1}(\Omega(\tau))) \cap L^\infty(0,T_0;W^{1,\infty}(\Omega(\tau)))$

$\cap W^{1,2}(0,T_0;H^{r+1}(\Omega(\tau))) \cap W^{1,\infty}(0,T_0;H^1(\Omega(\tau))),$

$S \in W^{1,\infty}(0,T_0);$

$\tilde{R}_2.$ $U \in \tilde{R}_1 \cap W^{1,\infty}(0,T_0;W^{1,\infty}(\Omega(\tau))) \cap W^{2,2}(0,T_0;L^2(\Omega(\tau))),$

$S \in \tilde{R}_1 \cap W^{2,2}(0,T_0);$

$\tilde{R}_3.$ $U \in \tilde{R}_2 \cap W^{2,2}(0,T_0;W^{1,\infty}(\Omega(\tau))) \cap W^{2,2}(0,T_0;H^2(\Omega(\tau)))$

$\cap W^{3,2}(0,T_0;L^2(\Omega(\tau))),$

$S \in \tilde{R}_2 \cap W^{3,2}(0,T_0).$

Let $\tilde{K}_2$ be a bound of the functions in all the norms of the spaces in $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}_3$.

We now introduce new co-ordinates x and t, connected with y and τ by the relation (2.3.1) and (2.3.2). If we write $U(y,\tau) = u(x,t)$ and $S(\tau) = s(t)$ in the transformed variable, then the problem (5.2.1) - (5.2.4) is transformed into the following coupled system of equations :

(5.2.5) $u_t - u_{xx} = -x\, u_x(1)u_x + s^2(t)f(u),\ x \in I,\ t \in (0,T]$,

(5.2.6) $u(x,0) = g(x),\ x \in I$,

(5.2.7) $u_x(0,t) = u(1,t) = 0,\ t > 0$

and

(5.2.8) $\frac{ds}{dt} = -u_x(1,t)s,\ t > 0$

with $s(0) = 1$.

The relationship of t with τ can be rewritten as (2.3.12).

Note that the regularity hypotheses $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}_3$ on the solution $\{U,S\}$ can be carried out to the solution $\{u,s\}$ and call these R_1, R_2 and R_3 respectively with a bound K_2 (say).

5.3 Galerkin Procedure and Some Approximation Lemmas.

Define $\overset{\circ}{H}{}^2(I)$ as in 2.4. If u and f(u) are sufficiently regular, then u satisfies the following weak form of (5.2.5)

(5.3.1) $(u_{tx}, v_x) + (u_{xx}, v_{xx}) = u_x(1)(xu_x, v_{xx}) - s^2(t)(f(u), v_{xx}),\ v \in \overset{\circ}{H}{}^2$

with $u(x,0) = g(x),\ x \in I$.

Let $\overset{\circ}{S}_h \subset \overset{\circ}{H}{}^2(I)$ be a finite dimensional subspace satisfying both approximation property (1.3.6) for $k = 2$ and the inverse property (1.3.10).

Now consider an H^1-Galerkin procedure for approximating the pair $\{u,s\}$ of (5.2.5) - (5.2.8). Denote the approximation of u by $u^h : (0,T] \to \overset{\circ}{S}_h$ and the approximation of s by s_h, where the pair $\{u^h, s_h\}$ is defined by the relations :

$$(5.3.2) \quad (u^h_{tx}, X_x)+(u^h_{xx}, X_{xx})=u^h_x(1)(xu^h_x, X_{xx})-s_h^2(f(u^h), X_{xx}),\ X \in \overset{\circ}{S}_h$$

with $$u^h(x,0) = Q_h g(x),$$

where Q_h is the appropriate projection onto $\overset{\circ}{S}_h$ and

$$(5.3.3) \qquad \frac{ds_h}{dt} = -u^h_x(1)s_h,$$

with $\quad s_h(0) = 1.$

Once the pair $\{u^h, s_h\}$ is known τ_h, the approximation of τ is given by (2.4.5).

Next, we define a fully discrete Galerkin procedure based on linearized modification of the backward differencing in time. Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$, and $t^n = n\Delta t, n=0,1,2,\dots,N$. Let $\varphi^n = \varphi(x,t^n)$ and $d_t \varphi^n = (\varphi^{n+1}-\varphi^n)/\Delta t$. Denote the approximation of u by $Z : \{t^0,t^1,\dots,t^N\} \to \overset{\circ}{S}_h$ and the approximation of s by W. Assuming that Z^n and W^n are known, we determine Z^{n+1} and W^{n+1} as follows :

$$(5.3.4) \quad (d_t Z^n_x, X_x)+(Z^{n+1}_{xx}, X_{xx})=Z^n_x(1)(xZ^{n+1}_x, X_{xx})-(W^n)^2(f(Z^n), X_{xx}),$$
$$X \in \overset{\circ}{S}_h$$

and

$$(5.3.5) \qquad d_t W^n = -Z_x^n(1)W^{n+1}.$$

Both these discretizations are $O(\Delta t)$ in time. A priori error estimates will be presented for $u^n - Z^n$ and $s^n - W^n$ in Section 5. Here we remark that (5.3.4) – (5.3.5) with given Z^o and W^o has solutions for sufficiently small Δt.

For higher accuracy in time, we shall consider the following variant of the Crank-Nicolson Galerkin scheme. Let $\varphi^{n+1/2} = (\frac{\varphi^{n+1}+\varphi^n}{2})$. Now replace (5.3.4) and (5.3.5) by

$$(5.3.6) \quad (d_t Z_x^n, X_x) + (Z_{xx}^{n+1/2}, X_{xx}) = E_\partial Z_x^n(1)(xZ_x^{n+1/2}, X_{xx}) - E(W^2)^n(f(EZ^n), X_{xx}), \ X \in \overset{o}{S}_h$$

and

$$(5.3.7) \quad d_t W^n = -Z_x^{n+1/2}(1)W^{n+1/2},$$

where $EZ^n = 3/2\ Z^n - 1/2\ Z^{n-1}$ and $E_\partial Z^n = 2Z^{n-1/2} - Z^{n-3/2} = \frac{2Z^n - 3Z^{n-1} + Z^{n-2}}{2}$.

The present scheme gives rise to a system of linear algebraic equations in Z^n and W^n $(n \geq 3)$ and is solvable for sufficiently small Δt, provided Z^o, Z^1, W^1 and Z^2 are given. We note that a starting procedure is needed to define Z^1, Z^2 as well as W^1, which will retain the overall accuracy of the method. A predictor-corrector version of (5.3.6) and (5.3.7) will suffice for our purpose.

The analysis proceeds, following Chapter 2, using an auxiliary projection. For this, consider the following bilinear form :

$$A(u;v,w) = (v_{xx},w_{xx})-u_x(1)(xv_x,w_{xx}), \text{ for } u \in W^{1,\infty};v,w \in \overset{\circ}{H}{}^2.$$

Let ρ be chosen sufficiently large that the bilinear form

$$A_\rho(u;v,w) = A(u;v,w) + \rho(v_x,w_x)$$

is coercive on $\overset{\circ}{H}{}^2(I)$ that is there is a positive constant α such that $A_\rho(u;v,v) \geq \alpha||v_{xx}||^2$. Let $\tilde{u} \in \overset{\circ}{S}_h$ be the Galerkin approximation to u with respect to the form A_ρ :

(5.3.8) $$A_\rho(u;u-\tilde{u},X) = 0, X \in \overset{\circ}{S}_h,$$

for each $t \in (0,T]$. The existence and uniqueness of u in (5.3.8) can be easily shown using Lax-Milgram Theorem (Theorem 1.3.3). Let $\eta = u-\tilde{u}$. We now state the following estimates for η without proof. The proof can be obtained following 2.6 or 3.4.

<u>Lemma</u> 5.3.1. There exists a constant $K_3 = K_3(\alpha,\rho,K_0,K_2)$ such that for $2 \leq m \leq r+1$ and $j = 0,1,2$

(5.3.9) $$||\eta||_j \leq K_3\, h^{m-j}\, ||u||_m ;$$

(5.3.10) $$||\eta_t||_j \leq K_3\, h^{m-j}\, (||u||_m + ||u_t||_m);$$

(5.3.11) $$|\eta_x(1)| \leq K_3 h^{2(m-2)}\, ||u||_m.$$

Further, we need an estimate for $||\eta||_{W^{1,\infty}}$ for our future use.

Following Douglas, Dupont and Wahlbin [19], we prove the following Lemma.

Lemma 5.3.2. Suppose $u \in L^{\infty}(0,T;W^{r+1,\infty})$, then there is a constant $K_4 = K_4(\rho,\alpha,K_2,K_0;||u||_{L^{\infty}(W^{r+1,\infty})})$ such that for $2 \leq m \leq r+1$

(5.3.12) $$||\eta_x||_{L^{\infty}} \leq K_4 h^{(m-1)}.$$

Proof. For $u \in \overset{\circ}{H}{}^2(I)$, let $\hat{u} \in \overset{\circ}{S}_h$ be the Ritz-approximation defined by

$$(u_{xx}-\hat{u}_{xx}, \chi_{xx}) = 0, \; \chi \in \overset{\circ}{S}_h.$$

Then, for sufficiently smooth u,

$$||(u-\hat{u})_x||_{L^{\infty}} \leq K_4' \, h^{m-1}, \; 2 \leq m \leq r+1.$$

The difference $\varphi = \tilde{u}-\hat{u} \in \overset{\circ}{S}_h$ obeys

$$A_\rho(u;\varphi,\chi) = -u_x(1)(x(u-\hat{u})_x, \chi_{xx}) + \rho((u-\hat{u})_x, \chi_{xx}).$$

The choice $\chi = \varphi$ gives

$$\alpha||\varphi_{xx}||^2 \leq K(||u_x||_{L^{\infty}}, \rho) \; ||(u-\hat{u})_x||_{L^{\infty}} \; ||\varphi_{xx}||.$$

Since,

$$||\eta_x||_{L^{\infty}} \leq ||(u-\hat{u})_x||_{L^{\infty}} + ||\varphi_x||_{L^{\infty}}$$

and

$$||\varphi_x||_{L^{\infty}} \leq ||\varphi_{xx}|| \leq \alpha^{-1} K \; ||(u-\hat{u})_x||_{L^{\infty}},$$

the inequality (5.3.12) is proved.

We note that under the hypothesis R_1, (5.3.9) and (5.3.12), it is easy to show that there exists constant K_5 such that

(5.3.13) $$\|\tilde{u}_x\|_{L^\infty(L^\infty)}, \|\tilde{u}\|_{L^2(H^2)} \leq K_5.$$

5.4 A Priori Error Estimates for Continuous Time Galerkin Approximations.

In this Section, we first develop a priori error bounds for the error $e = (u-u^h)$ and $e_1 = (s-s_h)$. Let $\zeta = u^h-\tilde{u}$, $\eta = u-\tilde{u}$, then $e = \eta-\zeta$. In order to maintain a uniform degree of approximation, we define $Q_h g = \tilde{u}(x,0)$, where $\tilde{u}$ is the projection of u onto $\overset{\circ}{S}_h$ defined through (5.3.8). Thus $\zeta(x,0) \equiv 0$.

We now compare u^h and $\tilde{u}$, defined in (5.3.2) and (5.3.8) respectively. We make the following additional assumption on u^h. Assume that there exists a positive constant $K^*(\geq 2K_5)$ such that

(5.4.1) $$\|u_x^h\|_{L^\infty(L^\infty)} \leq K^*.$$

From this as a Corollary a bound for $\|s_h\|_{L^\infty(0,T)}$ can be found out, using (5.4.1).

Indeed from (5.3.3) we have,

$$|s_h(t)| \leq 1 + \int_0^t |u_x^h(1)|\ |s_h|\ dt'.$$

Using Gronwall's Lemma (Lemma 1.3.9) and (5.4.1), we get

(5.4.2) $$\|s_h\|_{L^\infty(0,T)} \leq \exp(K^*T) \leq \tilde{K}(K^*).$$

Theorem 5.4.1. Assume the regularity condition R_1 and (5.3.13), (5.4.1) to hold. Then there exists $K_6=K_6(\alpha,\rho,K_0,K_1,K_2,K_3,K_5,K^*)$ such that

(5.4.3) $$\|\zeta\|_{L^\infty(H^1)}+\|e_1\|_{L^\infty(0,T)}+\beta\|\zeta\|_{L^2(H^2)}\leq K_6h^{r+1}, \text{for } r\geq 3.$$

Further, if h is sufficiently small, then K_6 can be chosen independent of K^*.

Proof. From (5.3.2), (5.3.1) and (5.3.8), we have

(5.4.4) $$(\zeta_{tx},\chi_x)+A_\rho(u;\zeta,\chi)=(\eta_{tx},\chi_x)-\rho(\eta_x,\chi_x)+\rho(\zeta_x,\chi_x)+\eta_x(1)$$
$$(xu^h_x,\chi_{xx})-\zeta_x(1)(xu^h_x,\chi_{xx})+s^2(f(u),\chi_{xx})-s_h^2(f(u^h),\chi_{xx}).$$

Subtracting (5.3.3) from (5.2.8) and multiplying the resulting equation by e_1, we get

(5.4.5) $$\frac{1}{2}\frac{d}{dt}|e_1|^2=-[u^h_x(1)e_1^2+\eta_x(1)se_1-\zeta_x(1)se_1].$$

Setting $\chi=\zeta$ in (5.4.4) and combining with (5.4.5), it follows that

(5.4.6) $$\frac{1}{2}\frac{d}{dt}(\|\zeta_x\|^2+|e_1|^2)+A_\rho(u;\zeta,\zeta)=(\eta_{tx},\zeta_x)-\rho(\eta_x,\zeta_x)+\rho\|\zeta_x\|^2$$
$$+\eta_x(1)(xu^h_x,\zeta_{xx})-\zeta_x(1)(xu^h_x,\zeta_{xx})+(s^2-s_h^2)(f(u^h),$$
$$\zeta_{xx})+s^2(f(u)-f(u^h),\zeta_{xx})-u^h_x(1)|e_1|^2-\eta_x(1)se_1$$
$$+\zeta_x(1)se_1.$$

Integrating by parts with respect to x the first two terms on

the right hand side of (5.4.6) and applying Inequality I of Chapter 1 to all terms except the fifth and the last terms, we obtain

$$(5.4.7)\quad \frac{1}{2}\frac{d}{dt}(||\zeta_x||^2+|e_1|^2)+\alpha||\zeta_{xx}||^2 \leq 7\varepsilon||\zeta_{xx}||^2+K_7(\rho,K_1,K_2,K^*,\varepsilon)$$

$$(||\eta_t||^2+||\eta||^2+|\eta_x(1)|^2)+K_8(\rho,K_1,K_2,\tilde{K}(K^*),K^*,\varepsilon)$$

$$(||\zeta_x||^2+|e_1|^2)+|\zeta_x(1)(xu_x^h,\zeta_{xx})|+|\zeta_x(1)\ se_1|.$$

To estimate the last but one term on the right hand side of (5.4.7), we note that $|\zeta_x(1)| \leq \sqrt{2}||\zeta_x||^{1/2}\ ||\zeta_{xx}||^{1/2}$. Now an application of Young's inequality that is $ab \leq \frac{\delta^{-p}a^p}{p} + \frac{\delta^q b^q}{q}$, $\frac{1}{p}+\frac{1}{q} = 1$, $a,b > 0$, where δ is appropriately chosen, gives

$$(5.4.8)\quad |\zeta_x(1)(xu_x^h,\zeta_{xx})| \leq K^*\ |\zeta_x(1)|\ ||\zeta_{xx}||$$

$$\leq \sqrt{2}\ K^*\ ||\zeta_x||^{1/2}\ ||\zeta_{xx}||^{3/2}$$

$$\leq \frac{(K^*)^4}{\delta^4}\ ||\zeta_x||^2+3/4\ \delta^{4/3}||\zeta_{xx}||^2, \text{for } p=4, q=4/3.$$

We next use $|\zeta_x(1)| \leq ||\zeta_{xx}||$ to obtain a bound of the last term

$$(5.4.9)\qquad |\zeta_x(1)\ se_1| \leq \frac{K_2}{\varepsilon}\ |e_1|^2 + \varepsilon||\zeta_{xx}||^2.$$

Combining (5.4.7) - (5.4.9), we obtain

$$(5.4.10)\quad \frac{1}{2}\frac{d}{dt}\ (||\zeta_x||^2+|e_1|^2)+\alpha||\zeta_{xx}||^2 \leq (8\varepsilon+3/4\ \delta^{4/3})||\zeta_{xx}||^2$$

$$+K_7(\rho,K_1,K_2,K^*,\varepsilon)(||\eta_t||^2+||\eta||^2+|\eta_x(1)|^2)+K_9(\rho,K_1,K_2,$$
$$\tilde{K}(K^*),K^*,\varepsilon,\delta)(||\zeta_x||^2+|e_1|^2).$$

Now choosing ε and δ appropriately so that $(8\varepsilon + 3/4\ \delta^{4/3})$ is less than α and integrating with respect to 't' from 0 to t, we get

$$(5.4.11)\quad ||\zeta(t)||_1^2+|e_1(t)|^2+\tilde{\alpha}\int_0^t||\zeta||_2^2 dt' \leq 2K_7\int_0^t(||\eta_t||^2+||\eta||^2 + |\eta_x(1)|^2)\, dt'+2K_9\int_0^t(||\zeta||_1^2+|e_1|^2)\, dt',$$

where we used $||\zeta|| \leq ||\zeta_x||$ and $||\zeta_x|| \leq ||\zeta_{xx}||$ for all $\zeta \in \overset{o}{H}{}^2$. In view of the estimates(5.3.9), (5.3.10) for j = 0 and (5.3.11), Gronwall's Lemma (Lemma 1.3.9) yields

$$||\zeta(t)||_1^2+|e_1(t)|^2+\tilde{\alpha}\int_0^t||\zeta||_2^2 dt' \leq 2K_3K_7\exp(2K_7t)\{h^{2(r+1)}\int_0^t(||u||_{r+1}^2 + ||u_t||_{r+1}^2)dt'+h^{4(r-1)}\int_0^t||u||_{r+1}^2\, dt'\}.$$

Now, taking supremum over all $t \in (0,T)$, we get the required estimate (5.4.3) for $2(r-1) \geq r+1$ or equivalently, for $r \geq 3$. To complete our arguments, we must show that K_6 can be chosen independent of K^* for sufficiently small h. We use the inverse hypothesis (1.3.10), (5.4.3) and (5.3.13) and obtain

$$||u_x^h||_{L^\infty(L^\infty)} \leq ||\tilde{u}_x||_{L^\infty(L^\infty)} + ||\zeta_x||_{L^\infty(L^\infty)}$$
$$\leq K_5 + K_0 h^{-1/2}\, ||\zeta_x||_{L^\infty(L^2)}$$

$$\leq K_5 + K_o K_6\, h^{-1/2}\, h^{r+1}.$$

Clearly, for sufficiently small h

(5.4.12) $$||u_x^h||_{L^\infty(L^\infty)} \leq 2K_5 \leq K^*.$$

Thus, K_6 can be chosen independent of K^*.

From Theorem 5.4.1, Lemmas 5.3.1, 5.3.2 and the inverse hypothesis (1.3.10), we get the following Theorem.

Theorem 5.4.2. Let all the assumptions of the Theorem 5.4.1 hold along with the inverse hypothesis (1.3.10) for $\overset{o}{S}_h$. Then there is a constant $K_{10} = K_{10}(K_o, K_2, K_4, K_6)$ such that for $r \geq 3$,

(5.4.13) $$||e||_{L^\infty(0,T;H^j(I))} \leq K_{10}\, h^{r+1-j},\ j = 0,1,$$

and

(5.4.14) $$||e_1||_{L^\infty(0,T)} \leq K_{10} h^{r+1}.$$

If additionally, $u \in L^\infty(0,T;W^{r+1,\infty})$, then

(5.4.15) $$||e||_{L^\infty(0,T;W^{1,\infty}(I))} \leq K_{10} h^r.$$

Proof. Since $||\zeta||_{L^\infty(L^2)} \leq ||\zeta||_{L^\infty(H^1)}$, we have by (5.4.3) of Theorem 5.4.1, combined with (5.3.9) for $j = 0$ the estimate (5.4.13) for the case $j = 0$.

Similarly for $j = 1$, we conclude (5.4.13) from (5.4.3), (5.3.9) and the triangle inequality.

To complete the proof of the estimate (5.4.15), we invoke inverse hypothesis (1.3.10) for ζ and the relation (5.3.12).

Lastly, (5.4.14) follows directly from the estimate (5.4.3).

To prove an optimal order of convergence in H^2-norm, we state the following Theorem.

Theorem 5.4.3. Under identical hypotheses as in Theorem 5.4.2, there exists a constant $K_{11} = K_{11}(\alpha,\rho,K_0,K_1,K_2,K_3,K_5,K_6,K_{10})$ such that

$$\|e\|_{L^\infty(0,T;H^2(I))} \le K_{11}h^{r-1}. \tag{5.4.16}$$

Proof. If the choice $\chi = \zeta_t$ is made in (5.4.4), then

$$\begin{aligned}\|\zeta_{tx}\|^2+(\zeta_{xx},\zeta_{xxt}) &= u_x^h(1)(x\zeta_x,\zeta_{xxt})-e_x(1)(x\tilde{u}_x,\zeta_{xxt}) \\ &\quad + (\eta_{tx},\zeta_{tx})-\rho(\eta_x,\zeta_{xt})-[s_h^2(f(u^h),\zeta_{xxt})-s^2(f(u),\zeta_{xxt})] \\ &= I_1+I_2+I_3+I_4+I_5.\end{aligned} \tag{5.4.17}$$

Now for I_1, integrating by parts with respect to x we get

$$I_1 = u_x^h(1)\zeta_x(1)\zeta_{xt}(1)-u_x^h(1)((\zeta_x+x\zeta_{xx}),\zeta_{xt}).$$

Then using the inverse hypothesis (1.3.10), the estimates (5.4.3) and (5.4.1) we obtain

$$\begin{aligned}|I_1| &\le |u_x^h(1)|\ \|\zeta_x\|_{L^\infty} K_0h^{-1/2}\|\zeta_{tx}\|+|u_x^h(1)|(\|\zeta_x\|+ \\ &\qquad \|\zeta_{xx}\|)\ \|\zeta_{xt}\| \\ &\le K(K_0,K^*,\varepsilon)[h^{-1}\|\zeta_x\|^2_{L^\infty}+\|\zeta_x\|^2+\|\zeta_{xx}\|^2] \\ &\qquad + 3\varepsilon\|\zeta_{xt}\|^2\end{aligned} \tag{5.4.18}$$

$$\leq K(K_o,K_6,K^*,\varepsilon)h^{2r}+K(K^*,K_o,\varepsilon)||\zeta_{xx}||^2+3\varepsilon||\zeta_{xt}||^2,$$

For I_2, we integrate by parts with respect to x, and get

$$I_2 = -e_x(1)\tilde{u}_x(1)\ \zeta_{xt}(1)+e_x(1)(x\tilde{u}_{xx}+\tilde{u}_x,\zeta_{xt}).$$

So,

$$(5.4.19)\quad |I_2| \leq K_5||e||_{W^{1,\infty}}K_oh^{-1/2}||\zeta_{xt}||+2K_5||e||_{W^{1,\infty}}||\zeta_{xt}||$$

$$\leq K(K_o,K_5,\varepsilon)(h^{-1}||e||^2_{W^{1,\infty}}+||e||^2_{W^{1,\infty}})+2\varepsilon||\zeta_{xt}||^2$$

$$\leq K(K_o,K_5,K_{10},\varepsilon)h^{2(r-1/2)} + 2\varepsilon\ ||\zeta_{xt}||^2.$$

The following bounds for I_3 and I_4 are easily obtained

$$(5.4.20)\quad |I_3| + |I_4| \leq K(\rho,K_3,\varepsilon)\ h^{2r} + 2\varepsilon||\zeta_{xt}||^2.$$

Finally, I_5 can be written as

$$I_5 = s^2(f(u)-f(u^h),\zeta_{xxt}) - e_1(e_1-2s)(f(u^h),\ \zeta_{xxt})$$

$$= I_5' + I_5''.$$

Now integrating by parts I_5' takes the form

$$I_5' = -s^2((f_u(u)-f_u(u^h))u_x + f_u(u^h)\ e_x,\zeta_{xt}).$$

The bound of I_5' is then given by

$$(5.4.21)\qquad |I_5'| \leq K(K_1,K_2,\varepsilon)\ ||e||_1^2 + 2\varepsilon\ ||\zeta_{xt}||^2$$

$$\leq K(K_1,K_2,K_{10},\varepsilon)\ h^{2r} + 2\varepsilon\ ||\zeta_{xt}||^2,$$

Similarly,

$$I_5''= -e_1(e_1-2s)f(0)\zeta_{xt}(1)+e_1(e_1-2s)(f_u(u^h)u_x^h,\zeta_{xt}).$$

The bound for I_5'' is given as follows :

$$|I_5''| \le K_1K_2(|e_1|^2+|e_1|)K_0h^{-1/2}||\zeta_{xt}||+K_1K^*K_2(|e_1|^2+|e_1|)||\zeta_{xt}||$$

$$\le K(K_0,K_1,K_2,K^*;\varepsilon)\{h^{-1}(|e_1|^4+|e_1|^2)+|e_1|^4+|e_1|^2)\}$$

$$+ 4\varepsilon||\zeta_{xt}||^2 ;$$

Using the estimate (5.4.15), we have

$$(5.4.22) \qquad |I_5''| \le K(K_0,K_1,K_2,K_{10},K^*,\varepsilon)\ h^{2r+1} + 4\varepsilon||\zeta_{xt}||^2.$$

Combining (5.4.17) - (5.4.22) and choosing ε so that the term $||\zeta_{xt}||$ from both the sides get cancelled, we obtain

$$(5.4.23) \quad \frac{1}{2}\frac{d}{dt}||\zeta_{xx}||^2 \le K(K_0,K^*)||\zeta_{xx}||^2+K(\rho,K_0,K_1,K_2,K_5,K_6,K_{10},K^*)\ h^{2r-1}.$$

Now, an application of Gronwall's Lemma (Lemma 1.3.9), Lemma 5.3.1 and triangle inequality give the estimate (5.4.16).

But here the constant K_9 depends on K^*. An argument similar to the one in Theorem 5.4.1 shows that K_9 is independent of K^*, for sufficiently small h.

Now, the Galerkin approximations of $U(y,\tau)$ and $S(\tau)$ are defined by

$$(5.4.24) \quad \begin{aligned} U^h(y,\tau) &= u^h(x,t) \\ \text{and} \quad S_h(\tau) &= s_h(t) \quad \text{respectively,} \end{aligned}$$

where

$$(5.4.25) \quad \begin{aligned} y &= s_h x \\ \tau &= \tau_h(t) \end{aligned}$$

and s_h, τ_h are given by (5.3.3), (2.4.5).

Theorems 5.4.2 and 5.4.3 lead to the following result.

Theorem 5.4.4. Suppose that $\tilde{R}_1$ holds and that $\overset{o}{S}_h$ satisfies the inverse hypothesis (1.3.10). Then the following estimates hold

(5.4.26) $$||s-s_h||_{L^\infty(0,T_o)} = O(h^{r+1}),$$

(5.4.27) $$||\tau-\tau_h||_{L^\infty(0,T_o)} = O(h^{r+1}),$$

and

(5.4.28) $$|||U-U^h|||_{L^\infty(0,T_o;H^j(\tilde{\Omega}(\tau)))} = O(h^{r+1-j}),\ j = 0,1,2.$$

If in addition, $U \varepsilon L^\infty(0,T_o;W^{r+1,\infty}(\tilde{\Omega}(\tau)))$, then

(5.4.29) $$|||U-U^h|||_{L^\infty(0,T_o;W^{1,\infty}(\tilde{\Omega}(\tau)))} = O(h^r).$$

Here $|||\cdot|||_{L^\infty(0,T_o;W^{m,p}(\tilde{\Omega}(\tau)))}$ is defined by

(5.4.30) $$|||\varphi|||_{L^\infty(0,T_o;W^{m,p})(\tilde{\Omega}(\tau)))} = \sup_{0<\tau\leq T_o} \{||\varphi||_{W^{m,p}(\tilde{\Omega}(\tau))}\},$$

$p = 2$ or ∞ and $\tilde{\Omega}(\tau)$ is given in (2.7.24).

5.5 A Priori Estimates for the Discrete Time Galerkin Approximations.

In this Section, we first develop a priori bounds for the errors (u^n-Z^n) and (s^n-W^n) at discrete time levels t^n, where Z^n and W^n are defined by (5.3.4) and (5.3.5). The results, obtained will be similar to the corresponding estimates for the continuous case in Section 5.4. We assume here that the

regularity hypothesis R_2 of Section 3 holds. Let $e_1 = s-W$, $\zeta = Z-\tilde{u}$ and $e = u-Z = \eta-\zeta$.

Theorem 5.5.1. Let Z^o be determined such that

(5.5.1) $$||Z^o-g||_j \le K_{12}h^{r+1-j}, \quad j = 0,1,2.$$

There exist constants $K_{13} = K_{13}(\rho,\alpha,K_o,K_1,K_2,K_3,K_5)$ and $h_o > 0$ such that if $h \le h_o$, $\Delta t = o(h)$ and $r \ge 3$, then

(5.5.2) $$\sup_{t^n}(||\zeta_x||^2+|e_1|^2)+\beta\sum_{n=0}^{N}||\zeta^n_{xx}||^2 \le K_{13}\{(\Delta t)^2+h^{2(r+1)}+h^{4r-2}\}.$$

Proof. From (5.2.8) and (5.3.5), we have for $n = 0,1,\dots,N-1$,

(5.5.3) $$d_t e_1^n = d_t s^n - \frac{ds}{dt}\bigg|_{t=t^{n+1}} - [Z_x^{n+1}(1)e_1^{n+1}+s^{n+1}\ e_x^{n+1}(1)].$$

Next from (5.3.1) and (5.3.8) for $t = t^{n+1}$, we get for $n=0,1,\dots,N-1$,

(5.5.4) $$(d_t\tilde{u}^n_x,\chi_x)+A_\rho(u^n;\tilde{u}^{n+1},\chi)=-(d_t\eta^n_x,\chi_x)+(d_t u^n_x - \frac{\partial u_x}{\partial t}\bigg|_{t^{n+1}},\chi_x)$$
$$-(s^2)^{n+1}(f(u^{n+1}),\chi_{xx})+\rho(u_x^{n+1},\chi_x)+(u_x^{n+1}(1)-u_x^n(1))$$
$$(x\tilde{u}_x^{n+1},\chi_{xx}).$$

Subtracting (5.5.4) from (5.3.4), it follows that

(5.5.5) $$(d_t\zeta^n_x,\chi_x)+A_\rho(u^n;\zeta^{n+1},\chi)=(d_t\eta^n_x,\chi_x)-(d_t u^n_x-\frac{\partial u_x}{\partial t}\bigg|_{t^{n+1}},\chi_x)$$
$$-\eta^n_x(1)(xZ_x^{n+1},\chi_{xx})+\zeta^n_x(1)(xZ^{n+1}_x,\chi_{xx})+[(s^2)^{n+1}(f(u^{n+1}),$$
$$\chi_{xx})-(W^2)^n(f(Z^n),\chi_{xx})]-(u_x^{n+1}(1)-u_x^n(1))(x\tilde{u}_x^{n+1},\chi_{xx})$$
$$-\rho(\eta_x^{n+1},\chi_x)+\rho(\zeta_x^{n+1},\chi_x).$$

But,

$$(5.5.6)\quad (s^2)^{n+1}(f(u^{n+1}),\chi_{xx})-(W^2)^n(f(Z^n),\chi_{xx})$$

$$= (s^2)^{n+1}(f(u^{n+1})-f(u^n),\chi_{xx})+(s^2)^{n+1}(f(u^n)-f(Z^n),$$

$$\chi_{xx})+(s^{n+1}-s^n)(s^{n+1}+s^n)(f(Z^n),\chi_{xx})$$

$$+\ e_1^n(s^n+W^n)(f(Z^n),\chi_{xx}).$$

Substituting (5.5.6) in (5.5.5) with $\chi = \zeta^{n+1}$ and combining with (5.5.3) after multiplying e_1^{n+1}, we have for $n = 0,1,\cdots,N-1$,

$$(5.5.7)\quad (d_t\zeta_x^n,\zeta_x^{n+1})+\langle d_te_1^n,e_1^{n+1}\rangle + A_\rho(u^n;\zeta^{n+1},\zeta^{n+1})=(d_t\eta_x^n,\zeta_x^{n+1})$$

$$+(\frac{\partial u_x}{\partial t}\Big|_{t^{n+1}}-d_tu_x^n,\zeta_x^{n+1})-\rho(\eta_x^{n+1},\zeta_x^{n+1})+\rho||\zeta_x^{n+1}||^2$$

$$-\eta_x^n(1)(xZ_x^{n+1},\zeta_{xx})+\zeta_x^n(1)(xZ_x^{n+1},\zeta_{xx}^{n+1})+(s^2)^{n+1}$$

$$(f(u^{n+1})-f(u^n),\zeta_{xx}^{n+1})+(s^2)^{n+1}(f(u^n)-f(Z^n),\zeta_{xx}^{n+1})$$

$$+\Delta t d_ts^n(s^{n+1}+s^n)(f(Z^n),\zeta_{xx}^{n+1})+e_1^n(s^n+W^n)(f(Z^n),\zeta_{xx}^{n+1})$$

$$-\Delta t d_tu_x^n(1)(x\tilde{u}_x^{n+1},\zeta_{xx}^{n+1})+\langle d_ts^n-\frac{ds}{dt}\Big|_{t^{n+1}},e_1^{n+1}\rangle$$

$$-\langle Z_x^{n+1}(1)e_1^{n+1},e_1^{n+1}\rangle-\langle s^{n+1}\eta_x^{n+1}(1),e_1^{n+1}\rangle$$

$$+\ \langle s^{n+1}\ \zeta_x^{n+1}(1),\ e_1^{n+1}\rangle.$$

Here $\langle\varphi,\Psi\rangle = \varphi\Psi$. Now we estimate the left hand side of (5.5.7) by

$$(5.5.8) \quad (d_t\zeta^n_x, \zeta^{n+1}_{xx}) + \langle d_t e^n_1, e^{n+1}_1\rangle + A_\rho(u^n; \zeta^{n+1}, \zeta^{n+1})$$

$$\geq \frac{1}{2\Delta t}\{(||\zeta^{n+1}_x||^2 - ||\zeta^n_x||^2) + (|e^{n+1}_1|^2 - |e^n_1|^2)\}$$

$$+ \alpha||\zeta^{n+1}_{xx}||^2.$$

Next, we integrate the first three terms on the right hand side of (5.5.7) and obtain

$$(5.5.9) \quad |(d_t\eta^n_x, \zeta^{n+1}_x) + (\frac{\partial u_x}{\partial t}\Big|_{t^{n+1}} - d_t u^n_x, \zeta^{n+1}_x) + \rho(\eta^{n+1}_x, \zeta^{n+1}_x)|$$

$$= |-(d_t\eta^n + (\frac{\partial u}{\partial t}\Big|_{t^{n+1}} - d_t u^n) + \rho\,\eta^{n+1}, \zeta^{n+1}_{xx})|$$

$$\leq K(\rho; \varepsilon)(||d_t\eta^n||^2 + \sigma^2_n + ||\eta^{n+1}||^2) + 3\varepsilon||\zeta^{n+1}_{xx}||^2$$

where

$$(5.5.10) \quad \sigma^2_n = (\int_{t^n}^{t^{n+1}} ||\frac{\partial^2 u}{\partial t^2}(\cdot, t')||dt')^2 \leq \Delta t \int_{t^n}^{t^{n+1}} ||\frac{\partial^2 u}{\partial t^2}||^2\, dt'$$

by Schwarz's inequality. We note that,

$$(5.5.11) \quad \Delta t \sum_{n=0}^{N-1} \sigma^2_n \leq (\Delta t)^2 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} ||\frac{\partial^2 u}{\partial t^2}||^2\, dt'$$

$$\leq (\Delta t)^2\, ||\frac{\partial^2 u}{\partial t^2}||^2_{L^2(0,T;L^2)}.$$

Applying Inequality I of Chapter 1, we see that

$$(5.5.12) \quad |(W^n + s^n)e^n_1(f(Z^n), \zeta^{n+1}_{xx})| \leq K(K_1; \varepsilon)(|W^n|^2 + |s^n|^2)|e^n_1|^2$$

$$+ 2\varepsilon||\zeta^{n+1}_{xx}||^2.$$

Similarly, we have

$$(5.5.13)\quad |(s^2)^{n+1}\{(f(u^{n+1})-f(u^n),\zeta_{xx}^{n+1})+(f(u^n)-f(Z^n),\zeta_{xx}^{n+1})\}|$$
$$\leq K(K_1,K_2,\varepsilon)((\Delta t)^2||d_t u^n||^2+||\eta^n||^2+||\zeta^n||^2)$$
$$+\ 3\varepsilon||\zeta_{xx}^{n+1}||^2,$$

$$(5.5.14)\quad |(s^n+s^{n+1})\Delta t\ d_t s^n(f(Z^n),\zeta_{xx}^{n+1})|\leq K(K_1,K_2,\varepsilon)(\Delta t)^2|d_t s^n|^2$$
$$+\ \varepsilon||\zeta_{xx}^{n+1}||^2$$

and

$$(5.5.15)\quad |\Delta t\ d_t u_x^n(1)(x\tilde{u}_x^{n+1},\zeta_{xx}^{n+1})|\leq K(K_5,\varepsilon)(\Delta t)^2|d_t u_x^n(1)|^2$$
$$+\ \varepsilon\ ||\zeta_{xx}^{n+1}||^2.$$

Further, the following estimates are routinely deduced applying Inequality I of Chapter 1

$$(5.5.16)\quad |\eta_x^n(1)(xZ_x^{n+1},\zeta_{xx}^{n+1})|\leq K(\varepsilon)||Z_x^{n+1}||^2|\eta_x^n(1)|^2+\varepsilon||\zeta_{xx}^{n+1}||^2,$$

$$(5.5.17)\quad |\langle d_t s^n-\left.\frac{ds}{dt}\right|_{t^{n+1}},e_1^{n+1}\rangle-\langle Z_x^{n+1}(1)\ e_1^{n+1},e_1^{n+1}\rangle|$$
$$\leq(|Z_x^{n+1}(1)|+1)|e_1^{n+1}|^2+\sigma_{1,n}^2,$$

where

$$(5.5.18)\quad \sigma_{1,n}^2=\left(\int_{t^n}^{t^{n+1}}\left|\frac{d^2s}{dt^2}\right|dt'\right)^2\leq\Delta t\int_{t^n}^{t^{n+1}}\left|\frac{d^2s}{dt^2}\right|^2 dt'$$

and

$$(5.5.18')\quad |-\langle s^{n+1}\eta_x^{n+1}(1),e_1^{n+1}\rangle+\langle s^{n+1}\zeta_x^{n+1}(1),e_1^{n+1}\rangle|$$
$$\leq K_2(|\eta_x^{n+1}(1)|+|\zeta_x^{n+1}(1)|)|e_1^{n+1}|$$

$$\leq |\eta_x^{n+1}(1)|^2+K(K_2;\varepsilon)|e_1^{n+1}|^2+\varepsilon||\zeta_{xx}^{n+1}||^2.$$

Lastly, using $|\zeta_x^n(1)| \leq \sqrt{2}\ ||\zeta_x^n||^{1/2}||\zeta_{xx}^{n+1}||^{1/2}$ we estimate by generalized Young's inequality (Inequality III of Chapter 1)

$$(5.5.18)\quad |\zeta_x^n(1)(xZ_x^{n+1},\zeta_{xx}^{n+1})| \leq \sqrt{2}||Z_x^{n+1}||_{L^\infty}||\zeta_x^n||^{1/2}||\zeta_{xx}^n||^{1/2}$$

$$||\zeta_{xx}^{n+1}|| \leq \frac{1}{4\varepsilon^3}||Z_x^{n+1}||_{L^\infty}^4||\zeta_x^n||^2+\frac{\varepsilon}{2}||\zeta_{xx}^n||^2+\frac{\varepsilon}{2}||\zeta_{xx}^{n+1}||^2.$$

Multiplying each of the estimates (5.5.8) - (5.5.19) by Δt and summing up $n = 0,1,2,\ldots,J-1$, we obtain

$$(5.5.20)\quad \tfrac{1}{2}\{(||\zeta_x^J||^2-||\zeta_x^o||^2)+(|e_1^J|^2-|e_1^o|^2)\}+\alpha\Delta t\sum_{n=0}^{J-1}||\zeta_{xx}^{n+1}||^2$$

$$\leq \Delta t(12\varepsilon+\varepsilon/2)\sum_{n=0}^{J-1}||\zeta_{xx}^{n+1}||^2+\frac{\Delta t}{2}\varepsilon\sum_{n=0}^{J-1}||\zeta_{xx}^n||^2$$

$$+K(\rho;\varepsilon)\Delta t\sum_{n=0}^{N-1}(\sigma_n^2+\sigma_{1,n}^2)+K(\rho,K_1,K_2;\varepsilon)\Delta t\sum_{n=0}^{J-1}(||\eta^n||^2$$

$$+||d_t\eta^n||^2+||Z_x^{n+1}||_{L^\infty}^2|\eta_x^n(1)|^2+|\eta_x^{n+1}(1)|^2)+K(K_1,K_2,$$

$$K_5;\varepsilon)(\Delta t)^3\sum_{n=0}^{J-1}(||d_tu^n||^2+||d_ts^n||^2+|d_tu_x^n(1)|^2)$$

$$+K(K_1,K_2,\rho;\varepsilon)\Delta t\sum_{n=0}^{J-1}(1+||Z_x^{n+1}||_{L^\infty}^4)||\zeta_x^n||^2$$

$$+\rho\Delta t||\zeta_x^J||^2+K(K_1,K_2;\varepsilon)\Delta t\sum_{n=0}^{J-1}\{(|W^n|+1)^2+(1+$$

$$|Z_x^n(1)|)\}|e_1^n|^2+\Delta t\ K(K_2;\varepsilon)(|Z_x^J(1)|+1)|e_1^J|^2.$$

Assume further that

$$(5.5.21) \qquad \|Z_x^n\|_{L^\infty} \leq K^*, \quad \text{for } n = 0,1,\dots,J.$$

We shall use an induction argument to treat $|W^n|$. As an induction hypothesis, assume for sufficiently small h

$$(5.5.22) \qquad |W^n| \leq 2K_2 \text{ , for } n = 0,1,2,\dots,J-1.$$

Using (5.5.21) and (5.5.22) in (5.5.20), we see that if for sufficiently small Δt the coefficients of $|e_1^J|^2$ and $\|\zeta_x^J\|^2$ can be made less than 1/2 and similarly for an appropriately chosen ε, the first two terms in the right hand side can be made less than $\alpha\Delta t \sum_{n=0}^{J-1} \|\zeta_{xx}^{n+1}\|^2$, then we have

$$(5.5.23) \quad \|\zeta_x^J\|^2+|e_1^J|^2+\beta\Delta t\sum_{n=0}^{J}\|\zeta_{xx}^n\|^2 \leq K(\rho)(\Delta t)^2\Big(\Big\|\frac{\partial^2 u}{\partial t^2}\Big\|^2_{L^2(L^2)}$$

$$+\Big\|\frac{d^2 s}{dt^2}\Big\|^2_{L^2(0,T)}\Big)+K(\rho,K_1,K_2,K_3,K^*)(h^{2(r+1)}$$

$$+h^{2(2r-1)})+K(K_1,K_2,K_5)(\Delta t)^2\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L^\infty(L^\infty)}$$

$$+\Big\|\frac{ds}{dt}\Big\|^2_{L^\infty(0,T)}\Big)+K(\rho,K_1,K_2,K^*)\Delta t\sum_{n=0}^{J-1}(\|\zeta_x^n\|^2+|e_1^n|^2).$$

An application of discrete-Gronwall's Lemma (Lemma 1.3.10) gives us the estimate (5.5.2), which shows that the induction hypothesis (5.5.22) holds for $n = J$, for sufficiently small h. However, the constant involved in the estimate depends on K^*. By repeating the arguments similar to that of Section 4, we easily show that this constant is independent of K^*, for sufficiently small h and $\Delta t = o(h)$.

Finally, an application of triangle inequality together with Theorem 5.5.1, Lemma 5.3.1 and the fact that $\|\zeta\| \leq \|\zeta_x\|$ and $\|\zeta_x\| \leq \|\zeta_{xx}\|$, for $\zeta \in \overset{o}{H}{}^2$ yields the following Theorem.

<u>Theorem</u> 5.5.2. Let all the assumptions of Theorem 5.5.1 hold. Then there exist constants $K_{14} = K_{14}(K_2, K_3, K_{13})$ and $h_o > 0$ such that if $0 < h \leq h_o, \Delta t = o(h)$ and $r \geq 3$,

$$(5.5.24) \quad \sup_{t^n} (\|e\| + h\|e\|_1 + |e_1|) \leq K_{14}(\Delta t + h^{r+1}).$$

Further, choosing $\chi = \Delta t\, d_t\zeta^n$ in (5.5.5) and applying Theorems 5.5.1, 5.5.2 and Lemma 5.3.1, one can easily prove the following Theorem.

<u>Theorem</u> 5.5.3. Suppose that R_2 holds, $r \geq 3$ and that the space time discretizations satisfy the relation $\Delta t = o(h)$. Then there exists a constant $K_{15} = K_{15}(\rho, \alpha, K_o, K_i, i \leq 3, K_5, K_{14})$ such that for h sufficiently small the following holds :

$$(5.5.25) \qquad \sup_{t^n} \|e\|_2 \leq K_{15}(\Delta t + h^{r-1}).$$

We shall now present the error analysis for the extrapolated Crank-Nicolson Galerkin scheme.

<u>Theorem</u> 5.5.4. Suppose that the pair $\{u, s\}$ satisfies the regularity condition R_3 and (5.3.13) holds. Further, suppose that Z^o, Z^1, W^1 and Z^2 are chosen to satisfy

$$(5.5.26) \quad |e_1^1| + |e_1^2| + \|\zeta^1\|_1 + \|\zeta^2\|_1 + \Delta t(\|\zeta_{xx}^{1/2}\| + \|\zeta_{xx}^{3/2}\|)$$
$$\leq K_{16}\{(\Delta t)^2 + h^{r+1}\}.$$

Then there exists constant $K_{17} = K_{17}(\rho, \alpha, K_0, K_1, K_2, K_3, K_5, K_{16})$, independent of h and Δt such that for h sufficiently small and $\Delta t = o(h)$ we have

$$(5.5.27) \quad \sup_{t^n} (\|e\| + |e_1| + h\|e\|_1) \leq K_{17}\{(\Delta t)^2 + h^{r+1}\}, \quad r \geq 3.$$

Proof. Subtracting (5.3.7) from (5.2.8) for $t = t^{n+1/2}$ and multiplying the resulting equation by $e_1^{n+1/2}$, we get

$$(5.5.28) \quad \langle d_t e_1^n, e_1^{n+1/2}\rangle = \langle \sigma_{1,n}, e_1^{n+1/2}\rangle - \langle u_x(1, t^{n+1/2}) s(t^{n+1/2}) - Z_x^{n+1/2}(1) W^{n+1/2}, e_1^{n+1/2}\rangle,$$

where

$$(5.5.29) \qquad \sigma_{1,n} = d_t s^n - \left.\frac{ds}{dt}\right|_{t^{n+1/2}}.$$

From (5.3.6) and (5.3.1), (5.3.8) for $t = t^{n+1/2}$, we obtain for $n \geq 2$

$$(5.5.30) \quad (d_t \zeta_x^n, \chi_x) + (\zeta_{xx}^{n+1/2}, \chi_{xx}) = (d_t \eta_x^n, \chi_x) - (d_t u_x^n - \frac{\partial u_x}{\partial t}(t^{n+1/2}), \chi_x)$$
$$- (u_{xx}^{n+1/2} - u_{xx}(t^{n+1/2}), \chi_{xx}) - \rho(\eta_x^{n+1/2}, \chi_x)$$
$$+ [\{E_\partial Z_x^n(1)(xZ_x^{n+1/2}, \chi_{xx}) - u_x(1, t^{n+1/2})(xu_x(t^{n+1/2}),$$
$$\chi_{xx})\} + \tfrac{1}{2}\{u_x^n(1)(x\eta_x^n, \chi_{xx}) + u_x^{n+1}(1)(x\eta_x^{n+1}, \chi_{xx})\}]$$
$$+ \{s^2(t^{n+1/2})(f(u(t^{n+1/2})), \chi_{xx}) - E(W^2)^n(f(EZ^n), \chi_{xx})\}.$$

Integrating by parts first, second and fourth terms on the right hand side of (5.5.30) for $\chi = \zeta^{n+1/2}$ and combining with

(5.5.28), we get on summing for $n = 2, 3, \ldots, J-1$,

(5.5.31) $\sum_{n=2}^{J-1}\{(d_t\zeta_x^n, \zeta_x^{n+1/2})+\langle d_t e_1^n, e_1^{n+1/2}\rangle\}+\sum_{n=2}^{J-1}||\zeta_{xx}^{n+1/2}||^2=\sum_{n=2}^{J-1}\{-d_t$

$\zeta_{xx}^{n+1/2})+(d_t u^n-\frac{\partial u}{\partial t}(t^{n+1/2}), \zeta_{xx}^{n+1/2})+\rho(\eta^{n+1/2}, \zeta_{xx}^{n+1/2})\}$

$-\sum_{n=2}^{J-1}(u_{xx}^{n+1/2}-u_{xx}(t^{n+1/2}), \zeta_{xx}^{n+1/2})+\sum_{n=2}^{J-1}[\{E_\partial\ Z_x^n(1)(x\tilde{u}_x^{n+1/}$

$\zeta_{xx}^{n+1/2})-u_x(1,t^{n+1/2})(xu_x(t^{n+1/2}), \zeta_{xx}^{n+1/2})+E_\partial Z_x^n(1)$

$(x\zeta_x^{n+1/2}, \zeta_{xx}^{n+1/2})\}+\frac{1}{2}\{u_x^{n+1}(1)(x\eta_x^{n+1}, \zeta_{xx}^{n+1/2})+u_x^n(1)(x\eta_x^n$

$\zeta_{xx}^{n+1})\}]+\sum_{n=2}^{J-1}\{s^2(t^{n+1})(f(u(t^{n+1/2})), \zeta_{xx}^{n+1/2})-E(W^2)^n$

$(f(EW^n), \zeta_{xx}^{n+1/2})\}+\sum_{n=2}^{J-1}\langle\sigma_{1,n}, e_1^{n+1/2}\rangle-\sum_{n=2}^{J-1}\langle(u(1,t^{n+1/2})$

$s(t^{n+1/2})-Z_x^{n+1/2}(1)W^{n+1/2}), e_1^{n+1/2}\rangle$

$=\sum_{n=2}^{J-1}\{-(d_t\eta, \zeta_{xx}^{n+1/2})+(\sigma_{2,n}, \zeta_{xx}^{n+1/2})+\rho(\eta^{n+1/2}, \zeta_{xx}^{n+1/2})$

$-(\sigma_{3,n}, \zeta_{xx}^{n+1/2})\}$

$+ I_1 + I_2 + \sum_{n=2}^{J-1}\langle\sigma_{1,n}, e_1^{n+1/2}\rangle + I_3,$

where,

$$\sigma_{2,n} = d_t u^n - \frac{\partial u}{\partial t}(t^{n+1/2})$$

and

$$\sigma_{3,n} = u_{xx}^{n+1/2}-u_{xx}(t^{n+1/2}).$$

The first two terms on the left hand side of (5.5.31) become

$$(5.5.32)\quad \sum_{n=2}^{J-1}\{(d_t\zeta_x^n,\zeta_x^{n+1/2})+\langle d_t e_1^n, e_1^{n+1/2}\rangle\} = \frac{1}{2\Delta t}\{(||\zeta_x^J||^2-||\zeta_x^2||^2)$$

$$+(|e_1^J|^2-|e_1^2|^2)\}.$$

We now estimate the terms on the right hand side of (5.5.31).
For the first four terms

$$(5.5.33)\quad |\sum_{n=2}^{J-1}\{-(d_t\eta^n,\zeta_{xx}^{n+1/2})+(\sigma_{2,n},\zeta_{xx}^{n+1/2})+\rho(\eta^{n+1/2},\zeta_{xx}^{n+1/2})$$

$$-(\sigma_{3,n},\zeta_{xx}^{n+1/2})\}| \le 4\varepsilon \sum_{n=2}^{J-1}||\zeta_{xx}^{n+1/2}||^2+K(\rho,\varepsilon)\sum_{n=2}^{J-1}$$

$$(||d_t\eta^n||^2+||\sigma_{2,n}||^2+||\sigma_{3,n}||^2+||\eta^{n+1/2}||^2).$$

However, the following holds :

$$(5.5.34)\quad \Delta t\sum_{n=2}^{J-1}(||\sigma_{2,n}||^2+||\sigma_{3,n}||^2)$$

$$\le (\Delta t)^4\sum_{n=2}^{J-1}(||u_{ttt}||^2_{L^2(t^n,t^{n+1},L^2)}+||u_{xxtt}||^2_{L^2(t^n,t^{n+1},L^2}$$

In a similar way, we obtain

$$(5.5.35)\quad |\sum_{n=2}^{J-1}\langle\sigma_{1,n},e_1^{n+1/2}\rangle| \le \sum_{n=2}^{J-1}|\sigma_{1,n}|^2+\sum_{n=2}^{J-1}|e_1^{n+1/2}|^2.$$

But,

$$(5.5.36)\quad \Delta t\sum_{n=2}^{J-1}|\sigma_{1,n}|^2 \le (\Delta t)^4\sum_{n=2}^{J-1}||s_{ttt}||^2_{L^2(t^n,t^{n+1})}.$$

Now we give an estimate of I_1 below

$$(5.5.37)\quad |I_1| = \Big|\sum_{n=2}^{J-1}\big[-E_\partial \eta_x^n(1)(x\tilde{u}_x^{n+1/2},\zeta_{xx}^{n+1/2})+E_\partial\, \zeta_x^n(1)(x\tilde{u}_x^{n+1/2},$$

$$\zeta_{xx}^{n+1/2})-(u_x(1,t^{n+1/2})-E_\partial u_x^n(1))(x\tilde{u}_x^{n+1/2},\zeta_{xx}^{n+1/2})$$

$$+u_x(1,t^{n+1/2})(x[u_x^{n+1/2}-u(x,t^{n+1/2})],\zeta_{xx}^{n+1/2})$$

$$+\tfrac{1}{2}\{(u_x^{n+1}(1)-u_x(1,t^{n+1/2})(x\eta_x^{n+1},\zeta_{xx}^{n+1/2})+(u_x^n(1)$$

$$-u_x(1,t^{n+1/2}))(x\eta_x^n,\zeta_{xx}^{n+1/2})\}+E_\partial Z_x^n(1)(x\zeta_x^{n+1/2},\zeta_{xx}^{n+1/2})$$

$$\leq 7\varepsilon\sum_{n=2}^{J-1}||\zeta_{xx}^{n+1/2}||^2+K(K_2,K_5,\varepsilon)\sum_{n=2}^{J-1}\{(|\eta^n(1)|^2+|\eta^{n-1}(1)|^2$$

$$+|\eta^{n-2}(1)|^2)+|\sigma_{4,n}|^2+|\sigma_{5,n}(1)|^2||\eta_x^n||^2+|\sigma_{6,n}(1)|^2$$

$$+||\sigma_{7,n}||^2\} + I_1' + I_1'' ,$$

where,

$$(5.5.38)\quad \sum_{n=2}^{J-1}|\sigma_{4,n}|^2=\sum_{n=2}^{J-1}|u_x(1,t^{n+1/2})-E_\partial u_x^n(1)|^2$$

$$\leq K(\Delta t)^3\ ||u_{tt}||^2_{L^2(0,T;W^{1,\infty})},$$

$$(5.5.39)\quad |\sigma_{5,n}(1)|^2=|u_x^{n+1}(1)-u_x(1,t^{n+1/2})|^2\leq K(\Delta t)^2||u_{xt}||^2_{L^\infty(L^\infty)},$$

$$(5.5.40)\quad |\sigma_{6,n}(1)|^2=|u_x^n(1)-u_x(1,t^{n+1/2})|^2\leq K(\Delta t)^2||u_{xt}||^2_{L^\infty(L^\infty)}\ ,$$

$$(5.5.41)\quad \sum_{n=2}^{J-1}||\sigma_{7,n}||^2=\sum_{n=2}^{J-1}||u_x^{n+1/2}-u_x(t^{n+1/2})||^2$$

$$\leq K(\Delta t)^3\ ||u_{tt}||^2_{L^2(H^1)}\ ,$$

$$(5.5.42)\quad I_1' = \Big|\sum_{n=2}^{J-1} E_\partial\, \zeta_x^n(1)(x\tilde{u}_x^{n+1/2}, \zeta_{xx}^{n+1/2})\Big|$$

and

$$(5.5.43)\quad I_1'' = \Big|\sum_{n=2}^{J-1} E_\partial Z_x^n(1)(x\zeta_x^{n+1/2}, \zeta_{xx}^{n+1/2})\Big|.$$

Further, we see that using $|\varphi_x(1)| \leq \sqrt{2}\, ||\varphi_x||^{1/2}||\varphi_{xx}||^{1/2}$, for $\varphi \in \overset{\circ}{H}{}^2$ I_1' can be bounded by

$$I_1' \leq K(K_5)\sum_{n=2}^{J-1}(2|\zeta_x^{n-1/2}(1)|+3|\zeta_x^{n-3/2}(1)|)||\zeta_{xx}^{n+1/2}||$$

$$\leq K(K_5)\sum_{n=2}^{J-1}(||\zeta_x^{n-1/2}||^{1/2}||\zeta_{xx}^{n-1/2}||^{1/2}+||\zeta_x^{n-3/2}||^{1/2}$$

$$||\zeta_{xx}^{n-3/2}||^{1/2})||\zeta_{xx}^{n+1/2}||.$$

Applying generalized Young's inequality (Inequality III of Chapter 1), we get

$$(5.5.44)\quad I_1' \leq \varepsilon||\zeta_{xx}^{n+1/2}||^2+K(K_5,\varepsilon)\sum_{n=2}^{J-1}(||\zeta_x^{n-1/2}||^2+||\zeta_x^{n-3/2}||^2)$$

$$+\frac{\varepsilon}{2}\sum_{n=2}^{J-1}(||\zeta_{xx}^{n-1/2}||^2+||\zeta_{xx}^{n-3/2}||^2)\leq 2\varepsilon\sum_{n=2}^{J-1}||\zeta_{xx}^{n+1/2}||^2$$

$$+\varepsilon||\zeta_{xx}^{3/2}||^2+\varepsilon/2||\zeta_{xx}^{1/2}||^2+K(K_5,\varepsilon)\sum_{n=2}^{J-1}(||\zeta_x^n||^2+||\zeta_x^{n-1}||^2$$

$$+||\zeta_x^{n-2}||^2)\leq 2\varepsilon\sum_{n=2}^{J-1}||\zeta_{xx}^{n+1/2}||^2+\varepsilon(||\zeta_{xx}^{1/2}||^2+||\zeta_{xx}^{3/2}||^2)$$

$$+K(K_5,\varepsilon)\sum_{n=0}^{J-1}||\zeta_x^n||^2.$$

Using Inequality I of Chapter 1, we have the following

$$(5.5.45)\quad I_1''\leq K(\varepsilon)\sum_{n=2}^{J-1}|E_\partial Z_x^n(1)|^2||\zeta_x^{n+1/2}||^2+\varepsilon\sum_{n=2}^{J-1}||\zeta_{xx}^{n+1/2}||^2.$$

Further the following also holds :

$$s^2(t^{n+1/2})f(u(t^{n+1/2}))-E(W^2)^n f(EZ^n)=(s^2(t^{n+1/2})-E(s^2)^n)$$

$$f(u(t^{n+1/2}))+(E(s^2)^n-E(W^2)^n)f(EZ^n)+E(s^2)^n\{(f(Eu^n)$$

$$-f(EZ^n))+(f(u(t^{n+1/2})-f(Eu^n))\}.$$

Hence by the inequality I of Chapter 1, we have

$$(5.5.46)\quad |I_2|\leq K(K_1;\varepsilon)\sum_{n=2}^{J-1}(|\sigma_{8,n}|^2+|\sigma_{9,n}|^2)+K(K_1,K_2;\varepsilon)\sum_{n=2}^{J-1}(||E\eta^n||^2$$

$$+||E\zeta^n||^2+||\sigma_{10,n}||^2)+5\varepsilon\sum_{n=2}^{J-1}||\zeta_{xx}^{n+1/2}||^2,$$

where,

$$(5.5.47)\quad \sum_{n=2}^{J-1}|\sigma_{8,n}|^2=\sum_{n=2}^{J-1}|E(W^2)^n-E(s^2)^n|^2\leq\sum_{n=2}^{J-1}|E(W^n+s^n)|^2|Ee_1^n|^2,$$

$$(5.5.48)\quad \sum_{n=2}^{J-1}|\sigma_{9,n}|^2=\sum_{n=2}^{J-1}|E(s^2)^n-s^2(t^{n+1/2})|^2\leq K(\Delta t)^3||s_{tt}||^2_{L^2(0,T}$$

and

$$(5.5.49)\quad \sum_{n=2}^{J-1}||\sigma_{10,n}||^2=\sum_{n=2}^{J-1}||Eu^n-u(t^{n+1/2})||^2\leq K(\Delta t)^3||u_{tt}||^2_{L^2(L^2)}.$$

Further,

$$u(1,t^{n+1/2})s(t^{n+1/2})-Z_x^{n+1/2}(1)W^{n+1/2}=(u_x(1,t^{n+1/2})-u_x^{n+1/2}(1))$$

$$s(t^{n+1/2})+Z_x^{n+1/2}(1)\; e_1^{n+1/2}+Z_x^{n+1/2}(s(t^{n+1/2})-s^{n+1/2})$$

$$+\;e_x^{n+1/2}(1)\;s(t^{n+1/2})\;,$$

therefore using $|\zeta_x(1)| \le ||\zeta_{xx}||$ we get

$$(5.5.50) \quad |I_3| \le 2\sum_{n=2}^{J-1} |Z_x^{n+1/2}(1)|^2 |e_1^{n+1/2}|^2 + \varepsilon \sum_{n=2}^{J-1} ||\zeta_x^{n+1/2}||^2$$

$$+ \sum_{n=2}^{J-1} (|\eta_x^{n+1/2}(1)|^2 + |s(t^{n+1/2}) - s^{n+1/2}|^2 + |\sigma_{11,n}(1)|^2)$$

$$+ K(K_2, \varepsilon) \sum_{n=2}^{J-1} |e_1^{n+1/2}|^2,$$

where,

$$(5.5.51) \quad \sum_{n=2}^{J-1} |\sigma_{11,n}(1)|^2 = \sum_{n=2}^{J-1} |u_x^{n+1/2}(1) - u_x(1, t^{n+1/2})|^2$$

$$\le K(\Delta t)^3 ||u_{tt}||^2_{L^2(W^{1,\infty})}.$$

Note that

$$(5.5.52) \quad \sum_{n=2}^{J-1} |s(t^{n+1/2}) - s^{n+1/2}|^2 \le K(\Delta t)^3 \; ||s_{tt}||^2_{L^2(0,T)}.$$

Combining (5.5.31)-(5.5.52) and multiplying the resulting equation by Δt, we get

$$(5.5.53) \quad \tfrac{1}{2}(||\zeta_x^J||^2 + |e_1^J|^2) + \Delta t \sum_{n=2}^{J-1} ||\zeta_{xx}^{n+1/2}||^2 \le 20\varepsilon \; \Delta t \sum_{n=2}^{J-1} ||\zeta_{xx}^{n+1/2}||^2$$

$$+ K(\rho, K_1, K_2, K_5; \varepsilon) \Delta t \sum_{n=2}^{J-1} (||d_t \eta^n||^2 + ||\eta^{n+1/2}||^2$$

$$+ ||E\eta^n||^2 + |\eta_x^{n+1/2}(1)|^2 + |\eta_x^n(1)|^2 + |\eta_x^{n-1}(1)|^2$$

$$+ |\eta_x^{n-2}(1)|^2) + K(K_1, K_2, K_5; \varepsilon) \; \Delta t \sum_{n=2}^{J} (\Delta t)^2 \; ||\eta_x^n||^2$$

$$+ K(\rho, K_1, K_2, K_5; \varepsilon)(\Delta t)^4 + K(K_5; \varepsilon) \; \Delta t \sum_{n=0}^{J-1} ||\zeta_x^n||^2$$

$$+ K(\varepsilon) \; \Delta t \sum_{n=2}^{J-2} |E_\partial Z_x^n(1)|^2 (||\zeta_x^{n+1}||^2 + ||\zeta_x^n||^2)$$

$$+K(\varepsilon)\,\Delta t|E_\partial Z_x^{J-1}(1)|^2||\zeta_x^J||^2+K\,\Delta t\sum_{n=2}^{J-1}(1+|Z_x^{n-1/2}(1)|^2)\;|e_1^n|^2$$

$$+K\,\Delta t(1+|Z_x^{J-1/2}(1)|^2)|e_1^J|^2+K\,\Delta t\sum_{n=2}^{J-1}(1+|Z_x^{n+1/2}(1)|^2)\;|e_1^n|^2$$

$$+K(K_2)\,\Delta t\sum_{n=2}^{J-1}(|EW^n|^2+1)(|e_1^n|^2+|e_1^{n-1}|^2)+\varepsilon\,\Delta t(||\zeta_{xx}^{1/2}||^2$$

$$+\;||\zeta_{xx}^{3/2}||^2)+||\zeta_x^2||^2+|e_1^2|^2.$$

Now assume further that for any $K^* \geq 2K_5$

$$(5.5.54)\quad |W^{n-1}|;||Z_x^n||_{L^\infty} \leq K^*,\ \text{for } n = 1,2,\dots,J.$$

From Lemma 5.3.1 and the estimates (5.5.53), (5.5.26) and (5.5.54) it follows that for sufficiently small Δt, the coefficients of $||\zeta_x^J||^2$ and $|e_1^J|^2$ can be made less than 1/2 and similarly by choosing $\varepsilon = \frac{1}{20}$ we have

$$||\zeta_x^J||^2+|e_1^J|^2 \leq K(\rho,K_1,K_2,K_3,K_{16},K^*)\{h^{2(r+1)}+h^{2(2r-1)}+h^{2r}(\Delta t)^2$$
$$+(\Delta t)^4\}+K(K_1,K_2,K_5,K^*)\;\Delta t\sum_{n=0}^{J-1}(||\zeta_x^n||^2+|e_1^n|^2).$$

Since $\Delta t = o(h)$ and $r \geq 3$, an application of discrete Gronwall's Lemma (Lemma 1.3.10) gives

$$(5.5.55)\quad ||\zeta_x^J||^2+|e_1^J|^2 \leq K\,(\rho,K_1,K_2,K_3,K_{16},K^*)\{(\Delta t)^4+h^{2(r+1)}\}.$$

Following the technique used in Theorem 5.4.1, it is easy to show that the constant K involved in (5.5.55) is independent of K^* for sufficiently small h. Further, using $||\zeta|| \leq ||\zeta_x||$,

for $\zeta \in \overset{o}{H}{}^2$, Lemma 5.3.1 and triangle inequality, we obtain the desired result.

Remark. To prove (5.5.27), we need (5.5.26). For this purpose, a predictor-corrector version of (5.3.6) and (5.3.7) will suffice as a starting procedure of sufficient accuracy. Since the techniques are similar to those presented in Theorem 5.5.1 and are similar to those presented in Douglas et al [17], modified appropriately for H^1-Galerkin procedure we shall not present them here. Further choosing $\chi = \Delta t\, d_t\zeta^n$ in (5.5.30) applying Lemma 5.3.1 one gets estimate of e in H^2 norms at each discrete time level t^n.

We are now looking for a discrete-time Galerkin approximation of $\{U,S\}$, where $\{U,S\}$ is the solution of the original problem (5.2.1) - (5.2.4). The approximation $\{U_h^n, S_h^n\}$ of $\{U,S\}$ is given by

$$U_h^n \equiv U_h^n(y_n, \tau_n) = U_h(\tau_h^n) = Z(x,t^n)$$

(5.5.57) and

$$S_h^n \equiv S_h(\tau_h^n) = W(t^n),$$

where $y_h^n = x\, W^n$ and τ_h^n is given by either of the two relations

(5.5.58) $\quad d_t\tau_h^n = (W^{n+1})^2$

or

(5.5.59) $\quad d_t\, \tau_h^n = (W^2)^{n+1/2}.$

Here τ_h^n corresponds to t^n under the approximate transfermation.

Consequently, the time mesh in τ_h^n corresponding to a uniform mesh Δt in t will not in general be uniform.

Let $\Delta\tau_h^n = \tau_h^{n+1} - \tau_h^n$ and $\Delta\tau_h = \max_{0\le n\le N} \Delta\tau_h^n$.

Finally, we establish the following Theorem for the discrete-time error estimates $U^n - U_h^n$ and $S^n - S_h^n$, where $U^n \equiv U(\tau_h^n)$, $S^n \equiv S(\tau_h^n)$ and τ^n corresponds to t^n in the original transformation.

Theorem 5.5.5. Suppose that $\tilde{R}_2$ holds, $r \ge 3$ and that the free boundary S is bounded away from zero that is there is a positive constant ν such that $S \ge \nu > 0$ for all $\tau \in [0,T]$. Then the following estimate holds for a backward difference in time Galerkin approximation

$$(5.5.60) \quad \sup_{0\le n\le N} \{ \|U^n - U_h^n\|_{H^j(\tilde{\Omega}^n)} + |S^n - S_h^n| \} = O(\Delta t + h^{r+1-j}),\ j=0,1,2,$$

where the norm $\|\cdot\|_{H^j(\tilde{\Omega}^n)}$ is understood in the usual sense and

$$\tilde{\Omega}^n = (0, \min(S^n, S_h^n, S(\tau^n))).$$

Proof. Since $\sup_n |s^n - W^n| = O(\Delta t + h^{r+1})$ and $s^n \ge \nu$, there is an $\varepsilon > 0$ such that $W^n \ge \nu - \varepsilon = \nu_1 > 0$ (say), for $n=0,1,2,\ldots,N$. From (5.5.58) $\frac{\Delta\tau_h^n}{\Delta t} = (W^{n+1})^2$, we have $\Delta\tau_h^n \ge \nu_1^2 \Delta t$, for $n = 0,1,\ldots,N$, and hence $\Delta\tau_h \ge \nu_1^2 \Delta t$ that is $\Delta\tau_h = O(\Delta t)$. Now applying the arguments in Theorem 3.7.5 and Theorem 5.5.2 – 5.2.3 we get the estimates (5.5.60).

For an expolated Crank-Nicolson Galerkin scheme, we have the following results.

Theorem 5.5.6. Suppose that $\tilde{R}_3$ is valid along with the assumptions in Theorem 5.5.5. Then

$$(5.5.61)\quad \sup_n \{||U^n-U_h^n||_{H^j(\tilde{\Omega}^n)}+|S^n-S_h^n|\} = O\{(\Delta t)^2+h^{r+1-j}\},\ j=0,1$$

holds.

<u>Proof</u>. It is easy to see that $W^n \geq \nu_1 > 0$ and $\Delta\tau_h = O(\Delta t)$.

Now

$$|S^n-S_h^n| \leq |e_1^n| + K_2|\tau^n-\tau_h^n|$$

and

$$||U^n-U_h^n||_{H^j(\tilde{\Omega}^n)} \leq ||e^n||_j + K_2|\tau^n-\tau_h^n|,\ j = 0,1.$$

Thus,

$$(5.5.62)\quad \sup_n \{||U^n-U_h^n||_j+|S^n-S_h^n|\} \leq K(K_2,K_{17})\{(\Delta t)^2+h^{r+1-j} + \sup_n |\tau^n-\tau_h^n|\}.$$

To estimate $|\tau^n-\tau_h^n|$, we have from (2.3.12) for $t = t^{n+1/2}$ and (5.5.59)

$$d_t(\tau^n-\tau_h^n)=d_t\tau^n-\frac{d\tau}{dt}\Big|_{t^{n+1/2}}+(s^2)^{n+1/2}-(W^2)^{n+1/2}+s^2(t^{n+1/2})-(s^2)^{n+1/2}.$$

Since,

$$|d_t\ \tau^n - \tau_h^n| \geq \frac{1}{\Delta t}\ (|\tau^{n+1}-\tau_h^{n+1}| - |\tau^n - \tau_h^n|),$$

$$\sum_{n=1}^{N-1}(d_t\tau^n - \frac{d}{dt}\Big|_{t^{n+1/2}})| \leq K(K_2)\ (\Delta t)$$

and
$$\sum_{n=1}^{N-1} |s^2(t^{n+1/2}) - (s^2)^{n+1/2}| \leq K(K_2)\ \Delta t,$$

we have summing on n, $n = 1,2,\ldots,N$, using the estimate (5.5.27) and multiplying by Δt, the following estimates

$$(5.5.63) \quad |\tau^J-\tau_h^J| \leq K(K_2)\,\{(\Delta t)^2+h^{r+1}\} + |\tau^1-\tau_h^1|.$$

Assuming (5.5.26), $|\tau^1-\tau_h^1| = O\{(\Delta t)^2 + h^{r+1}\}$ and taking supremum over all J ($0 \leq J \leq N$) in (5.5.63) we get

$$(5.5.64) \qquad |\tau^n-\tau_h^n| \leq K(K_2)\,\{(\Delta t)^2 + h^{r+1}\}.$$

Now combining (5.5.64) and (5.5.62), we get the required results.

CHAPTER 6

EXTENSIONS AND REMARKS

6.1 Introduction.

In this concluding chapter, we make some informal observations regarding extensions of our analysis to more general problems such as the ablation problems and fluid filtration in porous media etc., and discuss various modifications which might improve upon our analysis in certain directions. More over, we present a critical assessment of the problems and the methods, used for solving them. Further, we present a short discussion on superconvergence phenomena.

6.2 Refinements and Generalizations.

Throughout the thesis, we deal with the free boundary problems either with homogeneous Neumann or, with Dirichlet boundary conditions. But the above analysis can be easily extended to the free boundary problems even with time dependent boundary conditions. For example, in Chapter 2 if the flux at $y = 0$ is given by

$$(6.2.1) \qquad U_y(0,\tau) = \varphi(\tau) \ , \ \tau > 0$$

then the corresponding condition (2.3.10) for u can be rewritten as

$$(6.2.2) \qquad u_x(0,t) = \varphi(\tau(t))s.$$

In this way the nonlinear parabolic equation (2.3.8) - (2.3.9) and (6.2.2) with $u(1) = 0$ is coupled with two ordinary differential equations (2.3.11) and (2.3.12). The error analysis follows the same lines as the corresponding time independent case with qualitatively the same bounds. Similar analysis is also valid for the free boundary problems with time dependent Dirichlet boundary conditions that is

$$U(0,\tau) = \varphi(\tau).$$

Let s_h denote the approximation of the free boundary, then the error $s-s_h$ is of the same order as $u_x(1) - u_x^h(1)$. Since $x = 1$ is always a knot, a superconvergence occurs. Following quasiprojection technique for H^1-Galerkin method for heat equation [15], we can achieve a higher order of convergence at the knots than the global error for nonlinear Stefan problems.

Next, a more general single phase Stefan problem :

$$U_\tau = \frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial U}{\partial y}\right) + b(y,\tau)\frac{\partial U}{\partial y} + c(y,\tau)U + f(y,\tau),$$

with either Neumann or Dirichlet boundary conditions, can be tackled using the approach of Chapter 5.

6.3 Ablation Problems.

In an ablation problem a solid is melted by a moving fluid. It is generally assumed that the diffusion coefficient is so large that $U_{yy} = 0$ in the liquid. Thus, the heat input $q(\tau)$ from fluid to the solid is prescribed on the unknown melting

boundary $y = S(\tau)$, giving rise to a one phase Stefan problem [38].

Problem $\tilde{P}$. Given $T > 0$ and $g(y)$ for $y \varepsilon I$ with $g_y(0) = g(1) = 0$. Find $\{U(y,\tau), S(\tau)\}$ such that $S(\tau) \geq \nu > 0$ for $0 < \tau \leq T$, ν being a constant,

(6.3.1) $$S(0) = 1,$$

(6.3.2) $$U_\tau - (a(y)U_y)_y = 0, \quad \tau \varepsilon (0,T] \text{ and } y \varepsilon \Omega(\tau), \text{ where}$$

$$\Omega(\tau) = \{y : 0 < y < S(\tau)\},$$

(6.3.3) $$U(y,0) = g(y) \quad \text{for } y \varepsilon I,$$

(6.3.4) $$\left.\begin{array}{l} U_y(0,\tau) = -f(\tau) \\ U(S(\tau),\tau) = 0 \end{array}\right\} \quad \text{for } 0 < \tau \leq T,$$

and

(6.3.5) $$\frac{dS}{d\tau} = -a(y)\, U_y\Big|_{y=S(\tau)} + q(\tau) \quad \text{for } 0 < \tau \leq T.$$

Here, we make the following assumptions on q, a, g and the solution $\{U, S\}$ and collectively call them 'condition B'.

(6.3.6) CONDITION B.

(i) The function $a(\cdot)$ is twice continuously differentiable and has a uniform bound K_1 such that

$$K_1 \geq \max(|a|, |a_y|, |a_{yy}|).$$

(ii) There is a positive constant α such that $a(\cdot) \geq \alpha$.

(iii) $q(\tau)$ is a bounded smooth function of τ such that $|q|, |q_\tau| \leq K_1$

(iv) The pair $\{U, S\}$ is determined uniquely and is sufficiently smooth.

For simplicity, we take $f(\tau) = 0$, although the case $f(\tau) > 0$ can be treated similarly.

We state explicitly the regularity hypothesis on the solution $\{U,S\}$ of (6.3.1) - (6.3.5) by $\tilde{R}_1$ as given in 5.2.

The Problem $\tilde{P}$ can easily be transformed into a Problem P with a fixed domain by a Landau-type transformation [38]

$$y = S(\tau)\,x \qquad \tau = t. \tag{6.3.7}$$

If we write $U(y,\tau) = u(x,t)$ and $S(\tau) = s(t)$, then by the routine calculations of 2.3 we get the following transformed Problem P.

Problem P : Find a pair $\{u,s\}$ satisfying

(6.3.8) $s^2u_t-(a(xs)u_x)_x = -a(s)u_x(1)xu_x+sq(t)xu_x, (x,t)\varepsilon I \times (0,T]$,

(6.3.9) $u(x,0) = g(x)$ for $x \varepsilon I$,

(6.3.10) $u_x(0,t) = u(1,t) = 0$ for $0 < t \leq T$

and

(6.3.11) $\frac{ds}{dt} = -a(s)u_x(1)s^{-1}+q(t)$ for $0 < t \leq T$

with

$$s(0) = 1.$$

Note that the Problem P involves a nonlinear parabolic problem (6.3.8) - (6.3.10), coupled with an initial value problem for ordinary differential equation (6.3.11). Further, under the transformation the regularity assumptions on $\{U,S\}$ are automatically carried over to $\{u,s\}$ and are denoted collectively by R_1

with a bound K_2 (say).

Weak formulation and Galerkin procedure (see Chapter 1). Let $\overset{\circ}{H}{}^2 = \{v \in H^2 : v_x(0) = v(1) = 0\}$. By weak solution of (6.3.8) – (6.3.10), we mean a function $u = u(x,t)$ such that for each fixed $t \in (0,T]$, $u(\cdot,t) \in \overset{\circ}{H}{}^2(I)$, $u(x,0) = g(x)$ and u satisfies the following equation

$$(6.3.12) \quad s^2(u_{tx}, v_x) + ((a(xs)u_x)_x, v_{xx}) = a(s)u_x(1)(xu_x, v_{xx}) - q(t)s(xu_x, v_{xx}), \quad v \in \overset{\circ}{H}{}^2(I).$$

For each of the values of h in $(0,1]$, let $\overset{\circ}{S}_h \subset \overset{\circ}{H}{}^2$ be a finite dimensional subspace satisfying the approximation property (1.3.6) for $k = 2$ and the inverse property (1.3.9). The Galerkin approximation $u^h : (0,T] \to \overset{\circ}{S}_h$ of u is defined by

$$(6.3.13) \quad s_h^2(u^h_{xt}, \chi_x) + ((a(xs_h)u^h_x)_x, \chi_{xx}) = a(s_h)u^h_x(1)(xu^h_x, \chi_{xx}) - q(t)s_h(xu^h_x, \chi_{xx}), \quad \chi \in \overset{\circ}{S}_h$$

with

$$u^h_x(x,0) = Q_h g(x),$$

where Q_h is an appropriate projection of g onto $\overset{\circ}{S}_h$ to be defined later and s_h, the Galerkin approximation of s is given by

$$(6.3.14) \quad \frac{ds_h}{dt} = -a(s_h)u^h_x(1)\, s_h^{-1} + q(t) \text{ with } s_h(0) = 1.$$

The error analysis follows the procedure adopted in Chapters 3-5 using an auxiliary projection. Let

$$(6.3.15) \quad A(s,u;v,w) = ((a(xs)v_x)_x, w_{xx}) - a(s)u_x(1)(xv_x, w_{xx})$$
$$+q(t)s(xv_x, w_{xx}) \text{ for } u \in W^{1,\infty}, \ v \text{ and } w \in H^2.$$

For the operator A, the boundedness and Gårding type inequality similar to Chapter 5 hold with the modification that the constants M and ρ depend on $|s|$ along with other terms.

Let $A_\rho(s,u;v,w) = A(s,u;v,w) + \rho(v_x, w_x)$. Then we have

$$(6.3.16) \quad A_\rho(s,u;v,v) \geq \tilde{\alpha} \, ||v_{xx}||^2 \text{ for } u \in W^{1,\infty} \text{ and } v \in \overset{o}{H}{}^2.$$

For each $t > 0$, we define $\tilde{u} \in \overset{o}{S}_h$ as the Galerkin approximation of u with respect to the form A_ρ :

$$(6.3.17) \quad A_\rho(s,u;u-\tilde{u},X) = 0 \text{ for } X \in \overset{o}{S}_h.$$

Since A_ρ satisfies both boundedness and coercivity properties, an application of Lax-Milgram Theorem (Theorem 1.3.3) gives unique solution $\tilde{u}$ of (6.3.17) :

Let $\eta = u-\tilde{u}$. Following the analysis of Chapter 2 or 4, one proves the following Lemma.

<u>Lemma</u> 6.3.1. There exists a constant $K_3 = K_3(M,\alpha,\rho,K_0,K_1$ and $K_2)$ such that for $j = 0,1,2$ and $2 \leq m \leq r+1$,

$$||\eta||_j \leq K_3 h^{m-j} ||v||_m ;$$
$$||\eta_t|| \leq K_3 \, h^{m-j}(||u_t||_m + ||u||_m)$$

and

$$|\eta_x(1)| \leq K_3 \, h^{2(m-2)} ||u||_m.$$

Assume that there exists a constant K_4 such that

(6.3.18) $$||\tilde{u}_{tx}||_{L^\infty(L^2)} \le K_4.$$

This follows from the previous Lemma.

<u>A priori estimates</u>. We assume that for positive ν_1

(6.3.19) $$s_h \ge \nu_1.$$

This is not a restriction, since $s(=S) \ge \nu$ and we shall subsequently see that

$$||s-s_h||_{L^\infty(0,T)} = O(h^{r+1}).$$

Further, we assume that there exists a constant K^* such that

(6.3.20) $$||u^h||_{L^\infty(H^2)} \le K^*, \text{ where } K^* \ge 2K_2.$$

We shall see later that for small h, this indeed holds good.

Now we define the projection Q_h as follows :

(6.3.21) $A_\rho(s,u;u-Q_hu,X) = 0$ for $t = 0$.

Clearly, $u^h(x,0) = \tilde{u}(x,0)$.

We use below the standard notations $e_1 = s-s_h$, $e = u-u^h$, $\zeta = u^h-\tilde{u}$ and $e = \eta-\zeta$.

<u>Theorem</u> 6.3.2. Suppose that the regularity condition R_1 and (6.3.18) - (6.3.20) hold. Further suppose that the projection Q_h is defined by (6.3.21). Then there is a constant K_5 depending on $M, \rho, \alpha, \nu, K_0, K_1, K_3, K_4$ and K^* such that for $r \ge 3$,

(6.3.22) $$\|\zeta\|_{L^\infty(H^1)} + \|e_1\|_{L^\infty(0,T)} + \|\zeta\|_{L^2(H^2)} \le K_5 h^{r+1}.$$

Proof. From (6.3.12), (6.3.13) and (6.3.17), we have

(6.3.23) $$s_h^2(\zeta_{tx}, X_x) + A_\rho(s, u; \zeta, X) = s^2(\eta_{tx}, X_x) - \rho(\eta_x, X_x) + \rho(\zeta_x, X_x)$$
$$+ (s^2 - s_h^2)(\tilde{u}_{tx}, X_x) + ((a(xs) - a(xs_h))u^h_{xx}, X_{xx}) + (s - s_h)$$
$$(a_x(sx)u^h_x, X_{xx}) + s((a_x(xs) - a_x(xs_h))u^h_x, X_{xx})$$
$$-a(s_h)\eta_x(1)(xu^h_x, X_{xx}) + (s - s_h)q(t)(xu^h_x, X_{xx})$$
$$-(a(s) - a(s_h))u_x(1)(xu^h_x, X_{xx}) + a(s_h)\zeta_x(1)(xu^h_x, X_{xx}).$$

Subtracting (6.3.14) from (6.3.11) and multiplying by e_1, we get

(6.3.24) $$\frac{1}{2}\frac{d}{dt}|e_1|^2 = a(s_h)e_1(\eta_x(1) - \zeta_x(1))s^{-1} + a(s_h)u^h_x(1)(ss_h)^{-1}\, e_1^2$$
$$+ (a(s) - a(s_h))\, u_x(1)\, s^{-1}\, e_1.$$

Setting $X = \zeta$ in (6.3.23), adding the resulting equation to (6.3.24) and following the analysis of Chapter 5 with the additional information that $s_h \ge \nu_1$, we obtain

$$\frac{1}{2}\frac{d}{dt}(s_h^2(t)\|\zeta_x\|^2 + |e_1|^2) + \tilde{\alpha}\|\zeta_{xx}\|^2 \le K_6(h^{2(r+1)} + \|\zeta_x\|^2 + |e_1|^2),$$

where K_6 depends, in particular upon K^*. Then applying Gronwall's Lemma (Lemma 1.3.9), the required result follows.

Corollary 6.3.3. Let all the assumptions of Theorem 6.3.2 hold. Then for $r \ge 3$,

(6.3.25) $$||\zeta||_{L^\infty(L^2)} \leq K_5 h^{r+1}.$$

Further, if $\overset{o}{S}_h$ satisfies the inverse property (1.3.9) then

(6.3.26) $$||\zeta||_{L^\infty(H^2)} \leq K_5 h^{r}.$$

The following result gives the estimates for e and e_1. Since the analysis follows exactly the pattern of proof in Chapter 5, we state only the Theorem without proof.

Theorem 6.3.4. There is a constant K_7 depending on K_5 and K_3 such that for $r \geq 3$

(6.3.27) $$||e||_{L^\infty(H^j)} \leq K_7 \; h^{r+1-j} \; , \; j = 0,1,2$$

and

(6.3.28) $$||e_1||_{L^\infty(0,T)} \leq K_7 h^{r+1}.$$

Besides, for sufficiently small h and $r \geq 3$,

$$||u^h||_{L^\infty(H^2)} \leq 2K_4 \leq K^*$$

and consequently, K_7 can be chosen independent of K^*.

Finally, the Galerkin approximations of $\{U,S\}$ are defined as in (2.7.8) with $y = s_h(t)x$ and $\tau = \tau_h$ where s_h and τ_h are given by (6.3.14) and (2.4.5) respectively.

Theorem 6.3.5. Suppose that the condition $\tilde{R}_1$ holds, then the following estimates are valid for $r \geq 3$,

(6.3.29) $$||S-S_h||_{L^\infty(0,T_o)} = O(h^{r+1}),$$

$$(6.3.30) \qquad |||U-U^h|||_{L^\infty(H^j(\tilde{\Omega}(\tau)))} = O(h^{r+1-j}), \; j = 0,1,2,$$

where $|||\varphi|||_{L^\infty(H^j(\tilde{\Omega}(\tau)))}$ is given by (2.7.23) - (2.7.24).

Remark (i) For optimality, the finite dimensional spaces used are C^1-splines of degree $r \geq 3$.

(ii) It is to be noted that a more general problem like the one in Vasilev [62], where the basic equation is of the form

$$U_\tau - a(U)U_{yy} = \tilde{f}$$

can be analysed by the method of the Chapters 3,5 coupled with the above technique.

6.4 Fluid Filtration in Porous Media.

In this Section, we consider a semidiscrete finite element approximation to a free boundary problem for the pressure head of a compressible fluid flowing through a homogeneous porous medium. By straightening the free boundary by a co-ordinate transformation, a continuous-time Galerkin procedure is discussed and a priori estimates are derived. Earlier Nitsche [51] initiated the study of error analysis for a linear problem, posed by Magenes [41 , Problem III], but the present one is an extension to the nonlinear version of it.

The mathematical model of the phenomenon can be formally stated as follows [6]:

Problem $\tilde{P}$. Given $T_0 > 0$, find $\{U(y,\tau),S(\tau)\}$ such that $S(\tau) > 0$, $0 < \tau \leq T_0$,

(6.4.1) $\quad S(0) = 1,$

(6.4.2) $\quad U_\tau-(a(U)U_y)_y = 0$ for $(y,\tau) \in \Omega(\tau) \times (0,T_0]$;

(6.4.3) $\quad U(y,0) = g(y)$ for $y \in I$;

(6.4.4) $\quad U_y(0,\tau) = -f(\tau)$ for $\tau > 0$;

(6.4.5) $\quad U(S(\tau),\tau) = S(\tau)$ for $\tau > 0$;

and

(6.4.6) $\quad S_\tau + a(U)U_y\Big|_{y=S(\tau)} = 0$ for $\tau > 0$.

Here $f(\tau) \geq 0$ and is a smooth function. For simplicity, let $f(\tau) = 0$, but for $f(\tau) > 0$, all our analysis can be carried out easily with appropriate modifications.

Assume the same condition $\tilde{R}_1$ and the same regularity hypothesis $\tilde{R}_2$ as in 2.2 for $a(\cdot)$, g and the solution $\{U,S\}$ of (6.4.1) – (6.4.6).

By a co-ordinate transformation (2.3.1) and (2.3.2), the Problem $\tilde{P}$ is reduced to

Problem P. If $U(y,\tau) = u(x,t)$ and $S(\tau) = s(t)$, then an easy calculation as in 2.3 shows that the pair $\{u,s\}$ satisfies

(6.4.7) $\quad u_t-(a(u)u_x)_x = -a(u(1))u_x(1)xu_x$ in $I \times (0,T]$;

(6.4.8) $u(x,0) = g(x),\ x \in I;$

(6.4.9) $u_x(0,t) = 0,\ t > 0;$

(6.4.10) $u(1,t) = s(t),\ t > 0$

and

(6.4.11) $s_t + a(u(1))\, u_x(1)s = 0,\ t > 0$

with $s(0) = 1,$

where T corresponds to T_o of τ. Moreover, t and τ are related by (2.3.12).

Remark. The transformed Problem P involves a nonlinear parabolic equation (6.4.7) - (6.4.10) in a fixed domain, coupled with two ordinary differential equations (6.4.11) and (2.3.12). Further, the regularity hypothesis $\tilde{R}_2$ is easily carried over to the solution $\{u,s\}$ and these regularity condition be termed 'condition R_2' with a bound K_2 (see 2.3).

Weak solution and Galerkin approximation. Let $\overset{o}{H}{}^2(I) = \{v \in H^2(I): v_x(0) = 0\}$. From the Problem P, we get $u_t(1) = -a(s)su_x(1)$. The weak solution u of (6.4.7) - (6.4.10) is given by

(6.4.12) $$(u_{tx}, v_x) + ((a(u)u_x)_x, v_{xx}) + sa(s)u_x(1)v_x(1)$$
$$= a(s)u_x(1)(xu_x, v_{xx}),\ v \in \overset{o}{H}{}^2.$$

Let $\overset{o}{S}_h \subset \overset{o}{H}{}^2$ be a finite dimensional space satisfying the approximation property (1.3.6), k = 2 and the inverse property (1.3.9). Now the Galerkin approximation $\{u^h, s_h\}$ of

$\{u, s\}$ is defined as follows : Find $\{u^h, s_h\}$ such that $u^h:(0,T]$ satisfies

$$(6.4.13)\quad (u^h_{tx}, \chi_x)+((a(u^h)u^h_x)_x, \chi_{xx}) + s_h a(s_h)u^h_x(1)\chi_x(1)$$

$$= a(s_h)u^h_x(1)(xu^h_x, \chi_{xx}),\ \chi \in \overset{o}{S}_h$$

with $u^h(x,0) = Q_h g$, where Q_h is an appropriate projection defined later and s_h satisfies

$$(6.4.14)\quad \frac{ds_h}{dt} = -a(s_h)u^h_x(1)s_h \text{ with } s_h(0) = 1.$$

Further τ_h, the approximation to τ is stated as in (2.4.5).

The error analysis proceeds, following an auxiliary projection as defined in Section 3. Here in the present case, $A_\rho(s,u;v,w)$ is given by

$$(6.4.15)\quad A_\rho(s,u;v,w) = ((a(u)v_x)_x, w_{xx})-a(s)u_x(1)(xv_x, w_{xx})$$

$$+ \rho(v_x, w_x),\ u \in W^{1,\infty};\ v,w \in \overset{o}{H}{}^2.$$

Following Chapter 3, an auxiliary projection as well as the related estimates for η can be easily given. The estimates ζ in H^1 and H^2-norms can be deduced using the analysis similar to the previous Chapters. But the only term, which requires special treatment is $s\, a(s_h)\, \zeta_x(1)\zeta_x(1)$. Applying $|\zeta_x(1)| \le ||\zeta_{xx}||$ and $|\zeta_x(1)| \le \sqrt{2}\, ||\zeta_x||^{1/2}||\zeta_{xx}||^{1/2}$ and using Young's inequality (Inequality II of Chapter 1) for $p = 4$,

$q = 4/3$ one gets error estimates in ζ. Now the basic estimates of e and e_1 read as follows :

Theorem 6.4.1. Let u, s and g be sufficiently smooth and Q_h be defined as (6.3.21) for $t = 0$. Further, let $\overset{o}{S}_h$ satisfy the inverse property (1.3.9). Then the following estimates

$$||e||_{L^\infty(H^1)} + h||e||_{L^\infty(H^2)} + ||e_1||_{L^\infty(0,T)} = O(h^r)$$

hold for $r \geq 2$.

Finally, the Galerkin approximations U^h and S_h of U and S respectively are defined similar to (2.7.28) and the main result concerning the errors in $U-U^h$, $S-S_h$ and $\tau - \tau_h$ can be stated as a theorem.

Theorem 6.4.2. Under the assumption of the previous theorem and the regularity condition $\tilde{R}_1$ and $\tilde{R}_2$, the following estimates hold for $r \geq 2$,

$$||S-S_h||_{L^\infty(0,T_o)} + ||\tau-\tau_h||_{L^\infty(0,T_o)} = O(h^r)$$

and

$$|||U-U^h|||_{L^\infty(0,T_o;H^j(\tilde{\Omega}(\tau)))} = O(h^{r+1-j}), \ j = 1, 2,$$

where $|||\cdot|||_{L^\infty(0,T_o;H^j(\tilde{\Omega}(\tau)))}$ is defined as in (2.7.23)-(2.7.24).

Remark. Similar results can be derived for Dirichlet boundary conditions following the procedure of Chapter 3 and 5.

6.5 Remarks and an Overview.

In Chapters 2 and 3, optimality in H^1 and H^2 norms only is achieved, and H^2-estimate is established with the help of the inverse property (1.3.9). However, optimal orders of convergence in L^2 as well as H^1 and H^2-norms are derived in Chapters 4 and 5. For optimality in L^2, the finite dimensional spaces $\overset{o}{S}_h$ belong to a regular $S_h^{r,2}$ family (for a definition see Chapter 1) with at least $r \geq 3$. Here we note that H^1-Galerkin procedure suits well the Stefan problem with a nonlinear parabolic equation in non-divergence form, although higher regularity is assumed on the solution (see Chapter 4). We observe in Chapter 5 that unlike its proceding Chapters, we do not require an inverse property (1.3.9) and (1.3.10) is good enough for our error analysis.

In the Chapters 2, 3 and 4, we have discussed the global existence and uniqueness of the Galerkin approximations and consistently used a Landau type transformations in these Chapters to fix the domain, where as the time-scale transformation helps to decouple the resulting system. This helps to solve for u independent of s. This greatly simplifies our analysis. However, the transformed problem as given in Chapter 5 is a coupled one due to the nonzero source term. Even without a time scale transformation,the error estimates can be established following 6.3. This fact is useful in dealing with fully discrete Galerkin schemes.

We make the following observation regarding the symmetrization of the operator A in Chapter 2. Note that we are really not required to symmetrize since it does not bring any extra advantage. Rather in a certain sense, it complicates L^2 analysis for the auxiliary projection (see the difficulties regarding the choice of the domain of the formal adjoint accosicated with symmetrized operator A_ρ). The error analysis related to the auxiliary projection in Chapter 3 is different from that of Chapter 2 and is more general in its approach.

Another interesting observation about Chapter 3 is the following : Assuming the existence and uniqueness of the Galerkin approximation in a ball $||u^h||_{L^\infty(H^2)} \leq K$, for any $K > 0$, the error analysis has been derived and later to justify the above assumption the existence (global) and uniqueness has been proved in a neighbourhood of the exact solution. We believe, this argument can be carried out to study the error analysis for a nonlinear parabolic problem by H^1-Galerkin procedure.

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